OXFORD IB DIPLOMA PROGRAMME

IN COOPERATION WITH ib

EXAM PRACTICE

MATHEMATICS: ANALYSIS AND APPROACHES

HIGHER LEVEL

COURSE COMPANION

ENHANCED ONLINE

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Exam practice: chapters 1 – 4

1 P2: Give your answers to parts **a** to **e** to the nearest dollar.

When Maria turned 18 her grandparents gave her three options of how she might receive her birthday present.

Option A: She receives \$50 each month for three years.

Option B: She receives \$1 in the first month, \$4 in the second month, \$7 in the third month, increasing by \$3 each month for three years.

Option C: \$20 in the first month and increasing by 5% each month for three years.

a If Maria chooses Option A, calculate the total value of her present. (2)

b If Maria chooses Option B, calculate

i the amount of money she will receive in the 12th month

ii the total value of her present at the end of the three-year period. (4)

c If Maria chooses Option C, calculate

i the amount of money she would receive in the 12th month;

ii the total value of her present at the end of the three-year period. (4)

d State which of options A, B or C Maria should choose to give her the greatest total value of her present. (1)

2 P1: Given the polynomial $p(x) = 2x^4 - 9x^3 + 6x^2 + 11x - 6$

a State the value of

i the sum of the roots of $p(x)$

ii the product of the roots of $p(x)$. (2)

b Show that $p(x)$ is divisible by $x^2 - x - 2$ (4)

c Hence write $p(x)$ as a product of linear factors. (2)

3 P2: Two girls and four boys go to the cinema.

a Given that they sit in six consecutive seats in one row, find

i the number of different arrangements in the ways they can sit

ii the number of different arrangements in the ways they can sit if the girls are to sit together. (4)

b Determine the number of different ways they can select 3 of them to go to the foyer to buy popcorn for the group if at least one boy and at least one girl are to be selected.

4 P1: Consider the rational function $f(x) = \dfrac{2x}{3x-2}$, $x \neq \dfrac{2}{3}$.

a Show that f is self- inverse. (4)

b Hence, state the range of f. (1)

c State $\lim\limits_{x\to\infty} f(x)$. (1)

5 P2: A rock is thrown vertically upward from the surface of the moon at a velocity of v_0 m s^{-1}.

After t seconds its height is given by $h = 24t - 0.8t^2$ metres.

a Find expressions for the velocity and acceleration of the rock after t seconds. (3)

b Hence state

i the value of v_0

ii the acceleration due to gravity on the moon. (2)

c Find the maximum height the rock reaches and state the value of t when this occurs. (3)

d State how long it takes for the rock to fall back to the moon's surface. (2)

e Find the values of t for which the rock is at half of its maximum height. (3)

6 P1: For $n \in \mathbb{Z}$, simplify the following expressions as far as possible, giving your answers in the form $a + bi$.

a i^{4n} (2)

b i^{4n+1} (2)

c i^{4n+2} (2)

d i^{4n+3} (2)

7 P1: Find the exact values of $a, b \in \mathbb{R}$ for which $(a+bi)^2 = i$. You should clearly show your working and reasoning at each stage. (9)

8 P1: a Given that $z = 3 + i$ is one of the roots of $z^3 + az^2 + bz + 10 = 0$ find the other two roots. (3)

b Hence, find the values of the real constants a and b. (3)

9 P1: When the polynomial $p(x) = x^4 + ax^3 + bx^2 + 5x + 1$, (where a and b are real constants), is divided by $(x+1)$ the remainder is 2 .

When $p(x)$ is divided by $(x-2)$ the remainder is 71.

Find the remainder when $p(x)$ is divided by $(x-1)$. (9)

10 P2: The sum of the first n terms of an arithmetic sequence is given by $S_n = 2n^2 + 4n$.

a Write down the value of

i S_1

ii S_2. (2)

The n^{th} term of the arithmetic sequence is given by u_n.

b Find the value of u_2. (2)

c Find the common difference of the sequence. (2)

d Hence show that $u_n = 4n + 2$ for all $n \in \mathbb{N}$. (2)

e Find the least value of n for which $S_n > 100u_n$. (4)

11 P1: Consider the function defined by $f(x) = \begin{cases} \dfrac{-3-x}{x-1}, & x < 0 \\ x+3, & x \geq 0 \end{cases}$.

a Show that f is a continuous function at $x = 0$. (3)

b Show that f is not differentiable at $x = 0$. (4)

12 P1: Three consecutive terms of an infinite geometric sequence are $a+2,\ 6$ and $a+7$, where $a \in \mathbb{Z}$.

a **i** Write down two expression for the common ratio, r, in term of a.

ii Hence, show that a satisfies the equation $a^2 + 9a - 22 = 0$. (4)

b **i** Find the possible values of a .

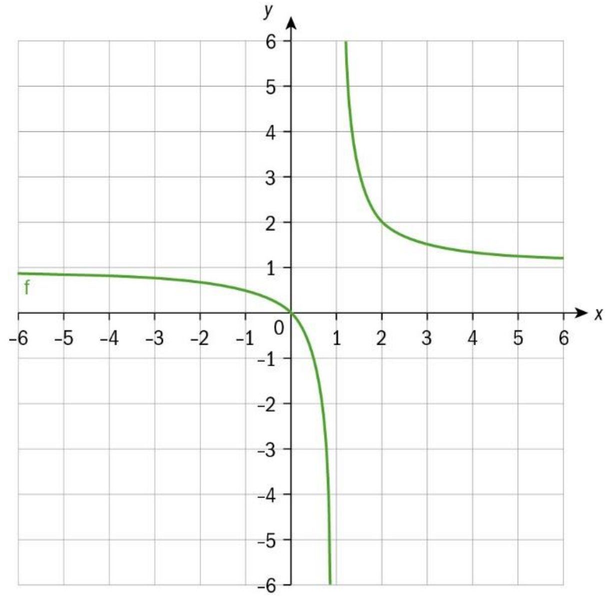
ii Find the possible values of r. (4)

The sum of all the terms of sequence is 9.

c State which value of r leads to this sum. Justify your answer. (2)

d Find the first term of the sequence. (2)

13 P1: The following diagram shows the graph of a function f defined for $x \neq 1$.



a State:

i the range of f

ii the equations of the asymptotes to the graph of f. (3)

b On the same axes, sketch and label

i the graph of $g(x) = f(x-2)+1$

ii the graph of $h(x) = \dfrac{1}{f(x)}$. (4)

c State the range and domain of

i $g(x)$

ii $h(x)$. (4)

14 P1: The diagram below shows two consecutive rows of Pascal Triangle.

1 **4** ***a*** ***b*** **1**

1 ***c*** **10** ***d*** ***e*** **1**

a Write down the values of a, b, c, d and e. (2)

b Find the coefficient of x^3 in the expansion $(2x-1)^5(x+1)^4$ (6)

15 P1: Let $f(x)=\frac{x+3}{5}$ and $g(x)=\frac{3x-1}{x-2}$.

a State the largest possible domain of g. (1)

b Find an expression for the inverse of g. (3)

c Hence

i determine $f\left(g^{-1}(x)\right)$

ii state the domain of $f\circ g^{-1}$. (3)

The line $y=k$ is an asymptote of the graph of g.

d Find the value of k. (2)

e By sketching graphs of $y=f(x)$ and $y=g(x)$, shade the region where $\left|\frac{3x-1}{x-2}\right|\geq\frac{x+3}{5}$.

Hence, write down solutions to the inequality. (6)

16 P1: Consider the rational function defined by $r(x)=\frac{3x-5}{x^2-3x+2}$ on S, where S is a subset of $\mathbb{R}$.

a Show that $r(x)$ can be written in the form $r(x)=\frac{A}{x-B}+\frac{C}{x-D}$, for $A,B,C,D\in\mathbb{Z}$, stating clearly the values of A,B,C and D. (5)

b Hence, state

i The largest possible domain S, of $r(x)$;

ii The equations of the asymptotes of $y=r(x)$. (4)

c Show that r is a decreasing function in all intervals of its domain S. (3)

d Justify that the range of r is $\mathbb{R}$. (4)

17 P1: Let s be the distance from the centre of the Earth of a falling meteorite. The velocity v of the meteorite is inversely proportional to $\sqrt{s}$.

a Find an expression for v in terms of s and an arbitrary constant k. (1)

b Show that the acceleration of the meteorite is inversely proportional to s^2. (4)

18 P1: Consider the sequence $\{u_n\}$ of positive terms defined iteratively by $u_n:\begin{cases}u_1=5\\ u_{n+1}=\frac{5u_n-4}{u_n}\end{cases}$.

a Find the first 3 terms of the sequence. Give your answers as fractions. (2)

b Prove by induction that $\{u_n\}$ is a decreasing sequence. (9)

c Prove by induction that $u_n > 4$ for all n. (7)

d Justify that $\{u_n\}$ is a convergent sequence and find the limit of this sequence. (5)

19 P3: In this question, you will investigate Descartes folium which is defined by the equation $x^3 + y^3 - axy = 0$, where a is a constant.

a Consider the case when $a = 9$.

i Show that this curve is not the graph of a function.

ii Show that $\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}$.

iii Hence determine the exact coordinates of the maximum point of the curve $y = y(x)$.

iv Find the coordinates of the points A and B on the curve for which there is no tangent defined. (19)

b Let $a > 0$ be a parameter that is to be determined.

i Show that the curve is symmetric with respect to the line $y = x$ for any $a > 0$.

ii Find the intersection points of the curve and the line $y = x$, in term of a.

iii Show that the equation of the curve can be written in the form $x + y = \frac{a}{\frac{x}{y} + \frac{y}{x} - 1}$

iv Hence, given that $y = -x - 1$ is an asymptote to the curve, determine the value of the parameter $a > 0$. (13)

20 P3: In this question you will investigate the function

$f(x) = kx(x-k)^2$, where $k > 0$.

a Find an expression for $f\left(\frac{2k}{3}\right)$ in terms of k. (2)

b Find and simplify an expression for $f'(x)$ in terms of k. (3)

c Hence, show that the graph of the function f has a local maximum at $x = \frac{k}{3}$. (2)

d Write down the coordinates of the local minimum of f. (2)

e Find the coordinates of the point of inflexion of f, in terms of k. (4)

f Sketch the graph of the function f for $0 \le x \le k$. On your sketch, mark the coordinates of the axes intercepts, the local maxima and minima, and the point of inflexion. (3)

g In terms of k, find the equation of the straight line L_k that contains the maximum and the minimum point of the graph of f. (5)

h Find the values of x for which $f(x)$ and L_k intersect. Hence prove that L_k contains the inflexion point of the graph of f. (6)

i Explain how the result proved in **h** can be used to determine the location of the inflexion point of the graph of f. Illustrate it using the graph sketched in part **f**.(3)

Answers

1	**a**	$50 \times 12 \times 3 = \$1800$	M1A1
	b	**i** $1 + 11 \times 3 = \$34$	M1A1
		ii $\frac{2 \times 1 + 35 \times 3}{2} \times 36 = \$1926$	M1A1
	c	**i** $20 \times 1.05^{11} = \$34$ (nearest dollar)	M1A1
		ii $\frac{1.05^{36} - 1}{1.05 - 1} \times 20 = \$1917$ (nearest dollar)	M1A1
	d	Option B	A1
2	**a**	**i** $S = \frac{9}{2}$	A1
		ii $P = \frac{-6}{2} = -3$	A1
	b	$(x^2 - x - 2)(ax^2 + bx + c) = 2x^4 - 9x^3 + 6x^2 + 11x - 6$	M1
		Equating coefficients	M1
		$a = 2$, $b = -7$ and $c = 3$	A1
		Checking consistency	
		$(x^2 - x - 2)(2x^2 - 7x + 3) = 2x^4 - 9x^3 + 6x^2 + 11x - 6$	R1
		(Could also be done by dividing the polynomials)	
	c	$p(x) = \underbrace{(x-2)(x+1)}_{(x^2-x-2)}\underbrace{(2x-1)(x-3)}_{(2x^2-7x+3)}$	A1A1
3	**a**	**i** $6! = 720$	M1A1
		ii $2 \times 2! \times 4! = 96$	M1A1
	b	$\binom{6}{3} - \binom{4}{3} = 20 - 4 = 16$	M1A1
4	**a**	$f(f(x)) = \dfrac{2\left(\frac{2x}{3x-2}\right)}{3\left(\frac{2x}{3x-2}\right) - 2}$	M1A1
		$= \dfrac{4x}{6x - 2(3x-2)} = x$	M1A1
		(Or alternatively, find f^{-1} directly)	

Hence f is a self-inverse | AG

b $y \neq \frac{2}{3}$ | A1

c $\lim\limits_{x\to\infty} f(x) = \frac{2}{3}$ | A1

5 a $v = 24 - 1.6t$ m s^{-1}, $a = -1.6$ m s^{-1} | M1A1A1

b **i** $v_0 = 24$ m s^{-1} | A1

ii 1.6 m s^{-2} | A1

c Use of GDC to find maximum | M1

180 m after 15 seconds | A1A1

d Use of GDC to find zero, or double the result from **c** | M1

30 seconds | A1

e Find intersection with line $h = 90$ | M1

4.39 s and 25.6 s | A1A1

6 a $i^{4n} = (i^4)^n = 1^n = 1 = 1 + 0i$ | M1A1

b $i^{4n+1} = i^{4n} \times i = i = 0 + i$ | M1A1

c $i^{4n+2} = i^{4n} \times i^2 = i^2 = -1 = -1 + 0i$ | M1A1

d $i^{4n+3} = i^{4n} \times i^3 = i^3 = -i = 0 - i$ | M1A1

7 $(a + bi)^2 = i \Rightarrow a^2 - b^2 + 2abi = 0 + 1i$ | M1A1

Equating real parts and imaginary parts gives | M1

$a^2 = b^2$, $2ab = 1$ | A1

Either $a = b$, which gives $2b^2 = 1 \Rightarrow b = \pm\frac{1}{\sqrt{2}}$ | M1A1

or $a = -b$, which gives $-2b^2 = 1$ (which is impossible as $b^2 \geq 0$) | R1

So $a = b = \frac{1}{\sqrt{2}}$ or $a = b = -\frac{1}{\sqrt{2}}$ | A1A1

8 a $z = 3 - i$ must be another root. | A1

Let the third root be α

Product of roots $= -10$ so $(3 - i)(3 + i)\alpha = -10 \Rightarrow \alpha = -1$ | M1A1

b $(z-(3+i))(z-(3-i))(z+1)=(z^2-6z+10)(z+1)=z^3-5z^2+4z+10$

$a=-5, b=4$ M1A1A1

9 $p(-1)=2 \Rightarrow 1-a+b-5+1=2 \Rightarrow -a+b=5$ M1A1

$p(2)=71 \Rightarrow 16+8a+4b+10+1=71 \Rightarrow 8a+4b=44 \Rightarrow 2a+b=11$ M1A1

Subtracting 1st equation from 2nd gives $3a=6$ (or other method) M1

$a=2, b=7$ A1A1

$p(1)=1+2+7+5+1=16$ M1A1

10 a **i** $S_1=6$ A1

ii $S_2=16$ A1

b $u_2=S_2-S_1$ M1

$u_2=10$ A1

c $u_1=S_1=6$ R1

$d=u_2-u_1=4$ A1

d $u_n=6+4(n-1)$ M1A1

$u_n=4n+2$ AG

e $S_n=6n+2(n-1)n$ A1

Using GDC to solve $6n+2(n-1)n>400n+200$ M1A1

gives $n=199$ A1

11 a $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(x+3)=3$ A1

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{-3-x}{x-1}=3$ A1

$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^-} f(x)=f(0)=3$ R1

Hence f is continuous at $x=0$ AG

b $f'(x)=\begin{cases}\frac{4}{(x-1)^2}, & x<0\\ 1, & x>0\end{cases}$ M1A1A1

$\lim\limits_{x\to 0^+} f'(x) \neq \lim\limits_{x\to 0^-} f'(x)$ R1

Hence f is not differentiable at $x=0$ AG

12 a **i** $\frac{6}{a+2}$ and $\frac{a+7}{6}$ A2

ii $\frac{6}{a+2}=\frac{a+7}{6}$ M1

$(a+7)(a+2)=36$ A1

$a^2+9a-22=0$ AG

b **i** Attempt to solve $a^2+9a-22=0$ M1

$a=2, a=-11$ A1

ii $r=-\frac{2}{3}, r=\frac{3}{2}$ A1A1

c $r=-\frac{2}{3}$. A1

The common ratio must have absolute value less than 1. R1

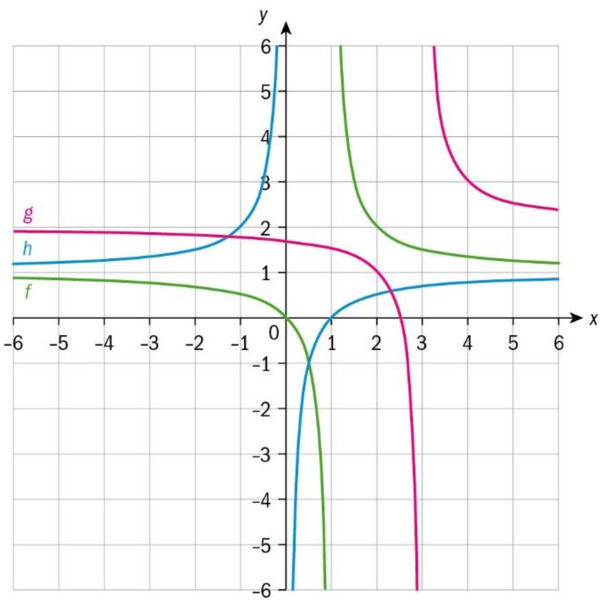
d $\frac{u_1}{1+\frac{2}{3}}=9$ M1

$u_1=15$ A1

13 a **i** $y\neq 1$ A1

ii $x=1, y=1$ A1A1

b



i for both branches of $g(x)$ A1A1

ii for both branches of $h(x)$ A1A1

c **i** Domain $x \neq 3$, range $y \neq 2$ A1A1

ii Domain $x \neq 0, (\text{and } x \neq 1, \text{ as 1 is not inthe domain of } f)$ A1

Range $y \neq 1, (\text{and } y \neq 0)$ A1

[note: Candidate is not penalised if they do not notice the absence of the point $(1,0)$ in the function $h(x)$.]

14 a $a = 6, b = 4$ A1

$c = e = 5, d = 10$ A1

b $\left({}^5C_3(2x)^3(-1)^2\right)\left({}^4C_0x^01^3\right)+\left({}^5C_2(2x)^2(-1)^3\right)\left({}^4C_1x^11^2\right)$ M1

$+\left({}^5C_1(2x)^1(-1)^4\right)\left({}^4C_2x^21^1\right)+\left({}^5C_0(2x)^0(-1)^5\right)\left({}^4C_3x^31^0\right)$

Coefficients are

$10\times2^3\times(-1)^2\times1+10\times2^2\times(-1)^3\times4+5\times2\times(-1)^4\times6+1\times(-1)^5\times4$ A1A1A1A1

$= -24$ A1

15 a $x \neq 2$ A1

b $x = \dfrac{3y-1}{y-2}$ M1

Solve for y M1

$g^{-1}(x) = \frac{2x-1}{x-3}$ A1

c **i** $f\left(g^{-1}(x)\right) = \dfrac{\frac{2x-1}{x-3}+3}{5}$ M1

$= \frac{5x-10}{5x-15} = \frac{x-2}{x-3}$ A1

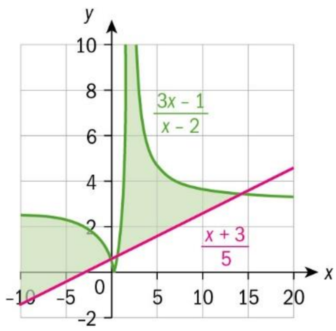
ii $x \neq 3$

A1

d $\lim\limits_{x\to+\infty} g(x) = \lim\limits_{x\to+\infty} \frac{3x-1}{x-2} = 3$ M1

$k = 3$ A1

e



Correct graph of *f* A1

Correct graph of *g* A2

$x \le -0.0711, \quad 0.660 \le x < 2, \quad 2 < x \le 14.1(3sf)$ A1A1A1

16 a $\frac{3x-5}{x^2-3x+2} = \frac{A}{x-1} + \frac{C}{x-2}$ M1

$B = 1$ and $D = 2$ A1

$A + C = 3$ and $-2A - C = -5$ M1

$A = 2$ and $C = 1$ A1A1

So $r(x) = \frac{2}{x-1} + \frac{1}{x-2}$

b **i** $S = \{x \mid x \neq 1, x \neq 2\}$ A1

ii $x = 1, x = 2, y = 0$ A1A1A1

c $r'(x) = -\dfrac{2}{(x-1)^2} - \dfrac{1}{(x-2)^2} < 0$. M1A1R1

So $r(x)$ is decreasing on S AG

d $\lim\limits_{x\to1^+} r(x) = \lim\limits_{x\to1^+}\left(\dfrac{2}{x-1} + \dfrac{1}{x-2}\right) = +\infty$ M1A1

$\lim\limits_{x\to2^-} r(x) = \lim\limits_{x\to2^-}\left(\dfrac{2}{x-1} + \dfrac{1}{x-2}\right) = -\infty$ A1

As r is continuous on the interval $(1,2)$ R1

range of r is $\mathbb{R}$. AG

17 a $v = \dfrac{k}{\sqrt{s}}$ A1

b $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(ks^{-\frac{1}{2}}\right) = -\dfrac{k}{2}s^{-\frac{3}{2}}\dfrac{\mathrm{d}s}{\mathrm{d}t}$ M1A1

Since $v = \dfrac{ds}{dt}$, it follows that

$a = -\dfrac{k}{2}s^{-\frac{3}{2}}ks^{-\frac{1}{2}} = \underbrace{-\dfrac{k^2}{2}}_{\text{constant}} \times \dfrac{1}{s^2}$ A1R1

So the acceleration of the meteorite is inversely proportional to s^2 . AG

18 a 5 and $\dfrac{21}{5}$ A1

$\dfrac{85}{21}$ A1

b Let $P(n)$ be the statement $u_{n+1} - u_n < 0$ M1

For $n = 1 \Rightarrow u_2 - u_1 = \dfrac{21}{5} - 5 < 0$ so $P(1)$ is true A1

Assume $P(k)$ is true $u_{k+1} - u_k < 0$ M1

Consider the statement for $n = k+1$ M1

$u_{k+2} - u_{k+1} = \dfrac{5u_{k+1} - 4}{u_{k+1}} - \dfrac{5u_k - 4}{u_k}$ M1A1

$= \dfrac{4}{u_k} - \dfrac{4}{u_{k+1}} = \dfrac{4}{u_k u_{k+1}}(u_{k+1} - u_k)$ which is < 0 since all the terms are positive M1A1

$P(1)$ is true and $P(k)$ true implies $P(k+1)$ true, so by the Principle of Mathematical Induction, $u_{n+1} - u_n < 0$ for all $n \in \mathbb{Z}^+$. R1

The function is decreasing AG

c Let $Q(n)$ be the statement $u_n > 4$ M1

$u_1 = 5$ so $Q(1)$ is true A1

Assume $Q(k)$ is true $u_k > 4$ M1

Consider the statement for $n = k + 1$ M1

$u_{k+1} = 5 - \frac{4}{u_k} > 5 - 1 = 4$ M1A1

$Q(1)$ is true and $Q(k)$ true implies $Q(k+1)$ true, so by the Principle of Mathematical Induction, $u_n > 4$ for all $n \in \mathbb{Z}^+$. R1

d $\{u_n\}$ is decreasing with a lower bound so must be convergent. R1

Let limit be l . It satisfies $l = \frac{5l - 4}{l}$ M1

$l^2 - 5l + 4 = (l - 4)(l - 1) = 0$ A1

So as all terms are > 4 limit must be 4. R1A1

19 a **i** For example $x = 1 \Rightarrow y^3 = 9y - 1$ which has 3 roots. M1A1

So it is not a function AG

ii $3x^2 + 3y^2 \frac{dy}{dx} - 9y - 9x \frac{dy}{dx} = 0$ M1A1A1

$(y^2 - 3x)\frac{dy}{dx} = 3y - x^2$ M1

$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}$ AG

iii $\frac{dy}{dx} = 0 \Rightarrow 3y - x^2 = 0 \Rightarrow y = \frac{x^2}{3}$ M1A1

$x^3 + \frac{x^6}{27} - 3x^3 = 0 \Rightarrow x^3(x^3 - 54) = 0$ M1

$x = 0, x = 3\sqrt[3]{2}$ A1

At $x = 0$ $\frac{dy}{dx}$ is not defined. R1

The only possible maximum is at $x = 3\sqrt[3]{2}$ R1

$y = \frac{x^2}{3} \Rightarrow y = \frac{(3\sqrt[3]{2})^2}{3} = 3\sqrt[3]{4}$ M1

Max is $\left(3\sqrt[3]{2}, 3\sqrt[3]{4}\right)$ A1

iv $y^2 - 3x = 0 \Rightarrow x = \dfrac{y^2}{3}$ M1

$\dfrac{y^6}{27} + y^3 - 3y^3 = 0 \Rightarrow y^3\left(y^3 - 54\right) = 0$ M1

$y = 0, y = 3\sqrt[3]{2}$ A1

A(0,0) and B$\left(3\sqrt[3]{4}, 3\sqrt[3]{2}\right)$ A1A1

b **i** The equation $x^3 + y^3 - axy = 0$ remains the same if we swap x and y. R1

Hence the curve is symmetric with respect to $y = x$. AG

ii $y = x \Rightarrow 2x^3 - ax^2 = 0 \Rightarrow x^2(2x - a) = 0 \Rightarrow x = 0, \dfrac{a}{2}$ M1A1

Points are $(0,0)$, $\left(\dfrac{a}{2}, \dfrac{a}{2}\right)$ A1A1

iii $x + y = \dfrac{a}{\dfrac{x}{y} + \dfrac{y}{x} - 1} \Rightarrow (x + y)\left(\dfrac{x}{y} + \dfrac{y}{x} - 1\right) = a$ M1

$\Rightarrow \dfrac{x^2}{y} + y - x + x + \dfrac{y^2}{x} - y = a \Rightarrow x^3 + y^3 = axy$ M1A1

As required

iv The asymptote is perpendicular to $y = x$

By symmetry as $x \to \infty, y \to -\infty$.

$x^3 \simeq -y^3$, $\lim\limits_{x\to\infty} \dfrac{x}{y} = \lim\limits_{x\to\infty} \dfrac{y}{x} = -1$ R1A1

$\lim\limits_{x\to\infty} \dfrac{a}{\dfrac{x}{y} + \dfrac{y}{x} - 1} = \dfrac{a}{-3}$ M1A1

So asymptote of $x + y = -1$ gives $a = 3$ A1

20 a $f\left(\dfrac{2k}{3}\right) = \dfrac{2k^2}{3}\left(\dfrac{2k}{3} - k\right)^2 = \dfrac{2k^4}{27}$ M1A1

b $f'(x) = k(x - k)^2 + 2kx(x - k)$ M1A1

$f'(x) = k(x-k)(3x-k)$ A1

c $f'\left(\frac{k}{3}\right) = 0$ A1

$f'\left(\left(\frac{k}{3}\right)^-\right) > 0$ and $f'\left(\left(\frac{k}{3}\right)^+\right) < 0$ R1

f has a local maximum at $x = \frac{k}{3}$ AG

d $(k, 0)$ A1A1

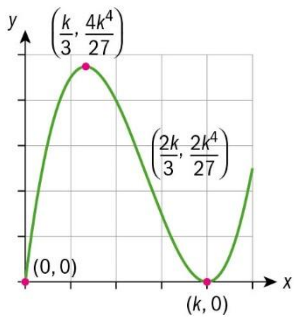
e $f''(x) = k(3x-k) + 3k(x-k)$ M1

$f''(x) = 2k(3x-2k)$ A1

$f''(x) = 0 \Rightarrow x = \frac{2k}{3}$. By inspection of the graph, f has a concavity change so this is a point of inflexion. A1

$\left(\frac{2k}{3}, \frac{2k^4}{27}\right)$ A1

f



Correct zeros A1

Correct shape with one maximum at $x = \frac{k}{3}$ and minimum at $x = k$ A1

Inflexion point at $x = \frac{2k}{3}$ A1

g Maximum at $\left(\frac{k}{3}, \frac{4k^4}{27}\right)$ A1

Minimum at $(k, 0)$

Gradient of line $L_k = \dfrac{0 - \frac{4k^4}{27}}{k - \frac{k}{3}} = -\frac{2k^3}{9}$ M1A1

Line $L_k : y = -\frac{2k^3}{9}(x-k)$ M1A1

h $-\frac{2k^3}{9}(x-k) = kx(x-k)^2$ M1

$k\left(x^2 - kx + \frac{2k^2}{9}\right)(x-k) = 0$ M1A1

$k\left(x - \frac{k}{3}\right)\left(x - \frac{2k}{3}\right)(x-k) = 0$

$x = k, x = \frac{k}{3}, x = \frac{2k}{3}$ M1A1

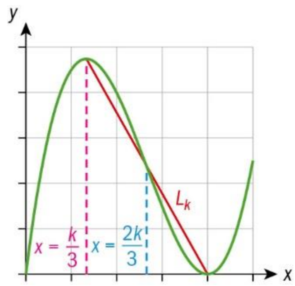
Therefore, L_k contains $\left(\frac{2k}{3}, \frac{2k^4}{27}\right)$ R1

Which is the inflexion point of the graph of f AG

i The inflexion point is at the intersection of the line L_k and f

and the vertical line $x = \frac{2k}{3}$ A1

It is the mid-point of the line section from the maximum to the minimum. R1



A1A1

Exam practice: chapters 1 – 6

1 P2: A study was conducted to determine if there was any correlation between a person's age (x) and their reaction time (T). A number of people were tested, and the mean reaction time calculated for each age is shown in the table below.

Age x (to the nearest 10 years)	10	20		30	40	60	70	80
Mean reaction time (T seconds)	0.125	0.148		0.166	0.221	0.231	0.270	0.341

a Find Pearson's product-moment correlation coefficient (r) for this data. (2)

b Use your result from part **a** to describe the strength of the correlation between x and T. (1)

c Find the equation of the regression line of T on x. (2)

d Using the above data, determine an estimate for the reaction time of a 50-year-old. (2)

e Explain why would it be unwise to use your answer to part **c** to determine the reaction time of a 90-year-old. (1)

2 P1: Without expanding any brackets, find the derivative of the following functions.

a $f(x) = (3x^2 + 1)(1 - 2x)$ (3)

b $g(x) = (x^2 + 3x)^2$ (2)

c $h(x) = \dfrac{4x - 1}{2 - x}$ (3)

3 P1: Consider the function $f(x) = kx(x-6)^2$, where $k > 0$.

a Find an expression for $f(2)$ in terms of k. (2)

b Find an expression for $f'(x)$ in terms of k. (3)

c Hence show that the graph of the function f has a local maximum at $x = 2$. (2)

Given that $f(3) = 54$,

d (i) Find the value of k

(ii) Write down the coordinates of the local maximum of f. (3)

e Sketch the graph of the function f for $0 \le x \le 7$. (2)

Let T be the tangent to the graph of the function f at $x = 3$.

f Find the gradient of T. (2)

The line L passes through the point (2, 10) and is perpendicular to T.

g Find an equation for L in the form $ax + by + c = 0,\ a, b, c \in \mathbb{Z}$. (3)

4 P1: Let $h = f \circ g$ for functions f and g, where $f(3) = 5$, $f'(3) = -2$, $g(2) = 3$ and $g'(2) = 4$.

a Show that h is decreasing at $x = 2$. (3)

b Find the equation of the tangent to the graph of h at $x = 2$. (3)

5 P2: Consider the family of rational functions defined by $r_k(x) = \dfrac{2kx}{x-k}$, $k \ne 0$.

a Find an expression for the inverse of r_k. (4)

The horizontal asymptote of the graph of $r_k(x)$ is $y = 4$.

b Find the value of k. (2)

c For the value of k found in part **b**, state the domain of the inverse of r_k. (1)

6 P1: Consider the function $f(x) = \dfrac{5-8x}{4x+3}$ $x \ne -\dfrac{3}{4}$, $x \in \mathbb{R}$.

a Write down the equations of the two asymptotes on the graph of $y = f(x)$. (2)

b State the range of f. (1)

c Find an expression for $(f \circ f)(x)$, giving your answer in the form $(f \circ f)(x) = \dfrac{ax+b}{cx+d}$.

State also the domain of $(f \circ f)(x)$. (5)

7 P1: A population of ferrets has mean age 5.25 years and standard deviation 1.2 years.

a Find the mean age of the same ferrets 3 years later. (2)

b Find the standard deviation of the same ferrets 2 years later. (2)

8 P2: The following raw data is a list the height of flowers (in cm) in Eve's garden:

26.5, 53.2, 27.5, 33.6, 44.6, 39.5, 24.9, 45.1, 47.8, 39.3, 33.1, 38.7, 44.1, 22.3, 44.1, 30.5, 25.5, 35.9, 37.1, 40.2, 23.3, 36.2, 34.8, 37.3

a Copy and complete the following grouped frequency table:

height, (x cm)	frequency
$20 \le x < 25$	
$25 \le x < 30$	
$30 \le x < 35$	
$35 \le x < 40$	
$40 \le x < 45$	
$45 \le x < 50$	
$50 \le x < 55$	

(3)

b Find an estimate for the mean height, using the frequency table (2)

c Find an estimate for the variance, using the frequency table (2)
d Find an estimate for the standard deviation, using the frequency table (2)
e Eve's neighbour's garden was also surveyed. It was found that the flowers in the neighbour's garden had a mean height of 32.1 cm and standard deviation 7.83 cm. Compare the heights of the flowers in both gardens, drawing specific conclusions. (3)

9 P2: Consider the function defined by $f(x) = \dfrac{x^2}{2x^3 - 1}$, $x \neq \sqrt[3]{\tfrac{1}{2}}$.

a Find an expression for $f'(x)$. (3)

b Find the equation of the tangent to the curve $y = f(x)$ at the point where $x = 1$. (4)

c Find the coordinates of the points on the curve $y = f(x)$ where the gradient is zero. (6)

d Without the use of technology, determine the range of values of x for which $f(x)$ is an increasing function.
To gain full marks, you must show all your working. (4)

10 P2: The first four terms of an arithmetic sequence are 80, 99, 118, 137.

a Find the 30th term in the sequence. (3)

b Find the sum to 13 terms. (2)
c Find the least number of terms required so that the sum exceeds 50 000.
To gain full marks, you must show all of your working. (5)

11 P2: Let $f(x) = \dfrac{3 + 7x}{(2 + 3x)(3 + 5x)}$ be a rational function.

a Express the above function in partial fractions. (5)

b Find the Binomial expansion of the above function up to and including the x^3 term. (4)

12 P1: Given that $f(x)=\dfrac{1}{x^2+1}$,

a Find f', the first derivative of f. (2)

b Prove that $(x^2+1)f'(x)+2xf(x)=0$. (3)

13 P2: Consider the function $f(x)=\dfrac{x^2+3}{x-2}$.

a Justify that the function f has no zeros. (1)

b Explain why the vertical asymptote of the curve of f is $x=2$. (1)

c Find the x - coordinates of the maximum and minimum of the curve of f . (7)

d Find the interval(s) where the curve of f is increasing. (2)

e Find the interval(s) where f is concave. (3)

14 P2: Two towers, A and B, are on opposite banks of a river. An observer O stands on the same bank as tower A. AOB is a triangle. The distance AO is 300 metres, the angle BAO is $63°$, and the angle BOA is $56°$. Find the distance between the two towers A and B to the nearest metre. (3)

15 P1: By applying the cosine rule, prove that in any triangle ABC the following identity holds: $\dfrac{\cos(A)}{a}+\dfrac{\cos(B)}{b}+\dfrac{\cos(C)}{c}=\dfrac{a^2+b^2+c^2}{2abc}$, where A, B, C are the angles of the triangle and a, b, c are the corresponding sides. You should clearly show all your working and state any formulae you use. (5)

16 P1: a The volume of a spherical balloon increases at a constant rate of π cm^3 per second.

Work out the rate at which the radius of the spherical balloon is increasing at the instant when the radius is $10cm$. (4)

b Work out the equation of the normal to the curve with implicit equation $3xy^2+3y-2=0$ at the point where the $y=-1$. (8)

17 P1: a Write down definitions for both an even function and an odd function. (2)

Consider the function $f(x)=\dfrac{7x+5}{x-3}$.

b Determine, with a justification, whether f is odd, even, or neither. (2)

c Determine, with a justification, whether f has an inverse function. If it does, find f^{-1}.

(4)

d Verify that $(f \circ f^{-1})(x) = x = (f^{-1} \circ f)(x)$, for every x in the domain of the f. (5)

18 P1: **a** Given the function $f(x) = \begin{cases} 2ax + b, x \leq 3 \\ ax + 3b, x > 3 \end{cases}$, find a, b such that $\lim_{x \to 3} f(x) = 10$. (4)

b State, with a reason, whether f is a continuous function. (2)

19 P1: Prove by induction that $A(n) = 7^{2n} + 16n - 1$ is a multiple of 64 for all $n \in \mathbb{N}$. (9)

20 P3: **a** Calculate the value of i^{2018}. (2)

b Find $z, w \in \mathbb{C}$ such that $\begin{matrix} 2z^* - 3w^* = 5 \\ z + iw = i \end{matrix}$. (7)

c Find the values of $\lambda \in \mathbb{R}$ such that the system of simultaneous equations

$\begin{matrix} 2x + \lambda y = \lambda - 2 \\ \lambda x + 2y = 4\lambda - 8 \end{matrix}$ has

i a unique solution, and find this solution

ii infinitely many solutions

iii no solutions. (12)

d Consider the equation $(x-1)(ax+2) - 3(x+2)(x-a) = 1$ for $a \in \mathbb{R}$.

i Show that the above equation is equivalent to $x^2(a-3) + x(2a-4) + 6a - 3 = 0$.

(3)

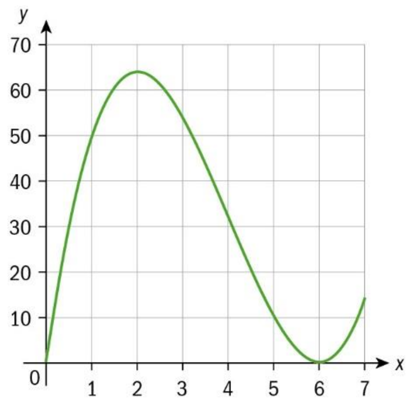
ii Given that this equation has one repeated root, find the possible values of the parameter $a \in \mathbb{R}$. (5)

Answers

1	**a**	Use of GDC to give	M1
		$r = 0.9675$	A1
	b	This is a strong positive correlation.	R1
	c	Use of GDC to give	M1
		$T = 0.00277x + 0.0919$	A1
	d	$T = 0.00277 \times 50 + 0.0919$	M1
		$= 0.230$ seconds	A1
	e	90 years lies outside the range of data we are given, which would therefore involve extrapolation of data.	R1
2	**a**	$f'(x) = \left(3x^2+1\right)'(1-2x) + \left(3x^2+1\right)(1-2x)'$	M1
		$f'(x) = 6x(1-2x) - 2\left(3x^2+1\right)$	A1
		$f'(x) = -18x^2 + 6x - 2$	A1
	b	$g'(x) = 2(2x+3)\left(x^2+3x\right)$	M1A1
	c	$h'(x) = \dfrac{(4x-1)'(2-x) - (4x-1)(2-x)'}{(2-x)^2}$	M1
		$h'(x) = \dfrac{4(2-x) + (4x-1)}{(2-x)^2}$	A1
		$h'(x) = \dfrac{7}{(2-x)^2}$	A1
3	**a**	$f(2) = 2k(2-6)^2 = 32k$	M1A1
	b	$f'(x) = k(x-6)^2 + 2kx(x-6)$	M1A1
		$f'(x) = k(x-6)(3x-6)$	A1
	c	$f'(2) = 0$	A1
		$f'\left(2^-\right) > 0$ and $f'\left(2^+\right) < 0$ (or considers second derivative)	R1
		f has a local maximum at $x = 2$	AG
	d	**i** $3k(3-6)^2 = 54 \Rightarrow k = 2$	M1A1

ii (2, 64) A1

e



Correct zeros A1

Correct shape with one maximum at $x = 2$ A1

f $f'(3) = 2(3-6)(3\times 3-6)$ M1

$f'(3) = -18$

A1

g $y - 10 = \frac{1}{18}(x-2)$ M1A1

$x - 18y + 178 = 0$ A1

4 a $h'(2) = f'(g(2))g'(2)$ M1

$= f'(3)\times 4 = -8$ A1

< 0 R1

h is decreasing at $x = 2$. AG

b $h(2) = f(g(2)) = f(3) = 5$ A1

$y - 5 = -8(x-2)$ M1

$y = -8x + 21$ A1

5 a $x = \frac{2ky}{y-k}$ M1

$xy - kx = 2ky \Rightarrow y = \frac{kx}{x-2k}$ M1A1

$r_k^{-1}(x) = \frac{kx}{x-2k}$ A1

b $\lim_{x\to\infty} r_k(x) = 4 \Rightarrow 2k = 4$ M1

$k = 2$ A1

c $x \neq 4$ A1

6 a $x = -\dfrac{3}{4}$ A1

$y = -2$ A1

b Range is $f(x) \neq -2$, $(f(x) \in \mathbb{R})$ A1

c $(f \circ f)(x) = \dfrac{5 - 8\left(\dfrac{5-8x}{4x+3}\right)}{4\left(\dfrac{5-8x}{4x+3}\right)+3}$ M1A1

$= \dfrac{5(4x+3) - 8(5-8x)}{4(5-8x)+3(4x+3)}$ A1

$= \dfrac{20x + 15 - 40 + 64x}{20 - 32x + 12x + 9}$

$= \dfrac{84x - 25}{29 - 20x}$ A1

$x \neq \dfrac{29}{20}$, $(x \in \mathbb{R})$ A1

7 a $5.25 + 3 = 8.25$ years M1A1

b 1.2 years M1A1

8 a

height, (x cm)	frequency
$20 \le x < 25$	4
$25 \le x < 30$	2
$30 \le x < 35$	4
$35 \le x < 40$	7
$40 \le x < 45$	4
$45 \le x < 50$	2
$50 \le x < 55$	1

M1A1A1

b Using GDC, mean height $= 35.6$ cm M1A1

c Using GDC, variance $= 68.3$ cm^2 M1A1

d Using GDC, SD $= 8.27$ cm M1A1

e On average, the neighbour's garden's flowers had a slightly lower height compared to Eve's. R1
The neighbour's flowers also had a smaller standard deviation, indicating they were grown to a more consistent length. R1R1

9 **a** Attempt to use quotient rule M1

$$f'(x) = \frac{2x(2x^3-1) - x^2(6x^2)}{(2x^3-1)^2} = \frac{-2x^4-2x}{(2x^3-1)^2}$$ A1A1

b At $x = 1$, $y = 1$ A1

$$f'(1) = \frac{-4}{1} = -4$$ A1

Equation is $y - 1 = -4(x-1)$ M1A1

Or $y = 5 - 4x$

c Setting $f'(x) = 0$ M1

$$\frac{-2x^4-2x}{(2x^3-1)^2} = 0$$

$-2x^4 - 2x = 0$ M1

$2x^4 + 2x = 0$

$2x(x^3+1) = 0$ M1

So $x = 0$ or $x = -1$ A1

Coordinates are $(0,0)$ and $\left(-1, -\frac{1}{3}\right)$ A1A1

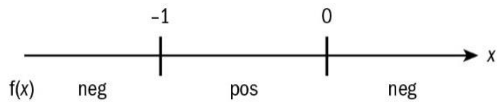
d Setting $f'(x) > 0$ M1

$-2x^4 - 2x > 0$

$2x^4 + 2x < 0$

$2x(x^3+1) < 0$

Critical values are $x = -1$, $x = 0$ A1



(or similar method to consider sign of $f(x)$ either side of critical values, or use of second derivative M1

So solution is $-1 < x < 0$ A1

10 a $u_{30} = u_1 + (n-1)d$ M1

$= 80 + 29 \times 19$

A1

$= 631$ A1

b $S_{13} = \frac{n}{2}[2u_1 + (n-1)d]$ M1

$= \frac{13}{2}[160 + 12 \times 19]$

$= 2522$ A1

c Require $S_n > 50\,000$

Consider $\frac{n}{2}[160 + 19(n-1)] = 50\,000$ M1

$19n^2 + 141n - 100\,000 = 0$ A1

Valid attempt to solve three term quadratic M1

$n = \frac{-141 \pm 2760.4}{38}$

$n > 0$, so $n = \frac{-141 + 2760.4}{38} = 68.9$ A1

So require 69 terms. A1

11 $\frac{3+7x}{(2+3x)(3+5x)} = \frac{A}{2+3x} + \frac{B}{3+5x}$ M1

Obtain the simultaneous equations: $\begin{array}{l} 3A + 2B = 3 \\ 5A + 3B = 7 \end{array}$ A1

Solve to give $A = 5$, $B = -6$ A2

$\frac{3+7x}{(2+3x)(3+5x)} = \frac{5}{2+3x} + \frac{-6}{3+5x}$ A1

$\frac{5}{2}(1+\frac{3}{2}x)^{-1} - \frac{6}{3}(1+\frac{5}{3}x)^{-1}$ M1

Expand first bracket: $\frac{5}{2}(1+\frac{3}{2}x)^{-1} = \frac{5}{2} - \frac{15}{4}x + \frac{45}{8}x^2 - \frac{135}{16}x^3$ M1

Expand second bracket: $-\frac{6}{3}(1+\frac{5}{3}x)^{-1} = -2 + \frac{10}{3}x - \frac{50}{9}x^2 + \frac{250}{3}x^3$ M1

$\frac{1}{2} - \frac{5}{12}x + \frac{5}{72}x^2 + \frac{3595}{48}x^3$ A1

12 a Proper use of quotient rule: $\frac{u'v - uv'}{v^2}$ M1

$f'(x) = \frac{-2x}{(x^2+1)^2}$ A1

b $(x^2+1)f'(x) + 2xf(x) = \frac{-2x(x^2+1)}{(x^2+1)^2} + \frac{2x}{(x^2+1)}$ M1A1

$= \frac{-2x^3 - 2x + 2x^3 + 2x}{(x^2+1)^2}$ A1

$= 0$ AG

13 a $x^2 + 3 \neq 0$ as $x^2 \geq 0$ for all x, R1

so $f(x)$ has no zeroes AG

b $f(x)$ is not defined at $x = 2$ as this would give denominator equal to 0 R1

Hence there is a vertical asymptote at so $f(x)$. AG

c $f'(x) = \frac{2x(x-2) - (x^2+3)}{(x-2)^2} = \frac{x^2 - 4x - 3}{(x-2)^2}$ M1A1

$f'(x) = 0 \Rightarrow x = 2 + \sqrt{7}$ or $x = 2 - \sqrt{7}$ A1

$f''(x) = \frac{14}{(x-2)^3}$ M1A1

$x = 2 + \sqrt{7}$ is minimum as $f''(2+\sqrt{7})$ is positive R1

$x = 2 - \sqrt{7}$ is maximum as $f''(2-\sqrt{7})$ is negative R1 (or alternatively, consideration of the signs of $f'(x)$ either side of the turning points)

d Attempt to solve the inequality $x^2 - 4x - 3 \geq 0$ M1

$x \leq 2 - \sqrt{7}$ or $x \geq 2 + \sqrt{7}$ A1

e Attempt to solve the inequality $\frac{14}{(x-2)^3} \leq 0$ R1

Equivalent to $(x-2)^3 \leq 0$ M1

$x \leq 2$ A1

14 Angle ABO is $61°$, angles on a triangle add up to $180°$ R1

Attempt to apply the Sine Rule: $\frac{AB}{\sin(56°)} = \frac{300}{\sin(61°)}$ M1

$AB = 284$ metres A1

15 Attempt to write the three formulae of the Cosine Rule: $a^2 = b^2 + c^2 - 2bc\cos(A)$

$b^2 = a^2 + c^2 - 2ac\cos(B)$

$c^2 = a^2 + b^2 - 2ab\cos(C)$ M1

Attempt to make cos the subject in the expressions:

$\cos(A) = \frac{a^2 - b^2 - c^2}{-2bc}$

$\cos(B) = \frac{b^2 - a^2 - c^2}{-2ac}$

$\cos(C) = \frac{c^2 - a^2 - b^2}{-2ab}$ A1

Substitute in the LHS: $\frac{a^2 - b^2 - c^2}{-2bc} + \frac{b^2 - a^2 - c^2}{-2ac} + \frac{c^2 - a^2 - b^2}{-2ab}$ M1

Add the fractions M1

RHS = $\frac{a^2 + b^2 + c^2}{2abc}$ A1

16 a Write volume of sphere as $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ M1A1

$\frac{dV}{dt} = \pi \Rightarrow \pi = 4\pi r^2 \frac{dr}{dt}$ M1

$\left.\frac{dr}{dt}\right|_{r=10} = \frac{1}{4\times 10^2} = \frac{1}{400}$ cm per second A1

b Differentiate implicitly: $3y^2 + 6xy\frac{dy}{dx} + 3\frac{dy}{dx} = 0$ M1A1

Make $\frac{dy}{dx}$ the subject: $\frac{dy}{dx} = \frac{-y^2}{1+2xy}$ M1A1

$y = -1 \Rightarrow x = \frac{5}{3}$ A1

Gradient of tangent at $\left(\frac{5}{3}, -1\right)$ is $\frac{dy}{dx} = \frac{3}{7}$ A1

gradient of the normal at $\left(\frac{5}{3}, -1\right)$ is $\frac{-7}{3}$ A1

normal is the line $y + 1 = \frac{-7}{3}(x - \frac{5}{3})$ A1

17 a Odd: $f(-x) = -f(x)$ A1

Even: $f(-x) = f(x)$ A1

b $f(-x) = \frac{-7x+5}{-x-3} \neq f(x)$, and $f(-x) = \frac{-7x+5}{-x-3} \neq -f(x)$ R1

The function is neither even, nor odd. A1

c f is a one-to-one function, so it has an inverse. R1A1

$y = \frac{7x+5}{x-3}$, make y the subject M1

$f^{-1}(x) = \frac{3x+5}{x-7}$ A1

d $\left(f \circ f^{-1}\right)(x) = \frac{7\left(\frac{3x+5}{x-7}\right)+5}{\left(\frac{3x+5}{x-7}\right)-3}$ M1

$= \frac{21x+35+5x-35}{3x+5-3x+21} = \frac{26x}{26}$ M1A1

$= x$ AG

Similarly, $\left(f^{-1} \circ f\right)(x) = \frac{3\left(\frac{7x+5}{x-3}\right)+5}{\left(\frac{7x+5}{x-3}\right)-7}$

$= \dfrac{21x+15+5x-15}{7x+5-7x+21} = \dfrac{26x}{26}$ M1A1

$= x$ AG

18 a For the limit $\lim_{x\to 3} f(x)$ to exist, we must have $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 10$. R1

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(2ax+b) = 6a+b = 10$ M1A1

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(ax+3b) = 3a+3b = 10$ A1

Attempt to solve the simultaneous equations: $\begin{matrix} 6a+b=10 \\ 3a+3b=10 \end{matrix}$ M1

$a = \dfrac{4}{3}, b = 2$ A1

b Need to check whether f is continuous at $x = 3$

Since $f(x)$ is defined at $x = 3$ and $\lim_{x\to 3} f(x)$ exists by part **a**, for f to be continuous we must have $\lim_{x\to 3} f(x) = f(3)$. R1

$f(3) = 2 \times \dfrac{4}{3} \times 3 + 2 = 10 = \lim_{x\to 3} f(x)$ M1A1

So f is a continuous function. A1

19 When $n=1$, $A(1) = 7^2+16-1 = 64$ so result holds. M1A1

Assume that $A(k) = 7^{2k}+16k-1$ is a multiple of 64, i.e. that $64a = 7^{2k}+16k-1$ for some $a \in \mathbb{Z}$ M1

Then $7^{2k} = 64a+1-16k$ A1

Now attempt to prove true for $A(k+1)$ M1

$A(k+1) = 7^{2k+2}+16(k+1)-1$ M1

$= 49\times 7^{2k}+16k+16-1$

$= 49(64a+1-16k)+16k+15$ A1

$= 64(49a+1-12k)$ which is a multiple of 64 A1R1

The result was true for $n=1$ and, when assumed true for $n=k$, it has been proved true for $n=k+1$. Therefore, byu mathematical induction, the result is true for all *n.*

20 a $(i^2)^{1009}$ M1

-1 A1

b Let $\begin{matrix} z = a + ib \\ w = c + id \end{matrix}$. Attempt to substitute into the system $\begin{matrix} 2z^* - 3w^* = 5 \\ z + iw = i \end{matrix}$. M1

Equating real parts and imaginary parts gives the system of equations: $\begin{matrix} 2a - 3c = 5 \\ -2b + 3d = 0 \\ a - d = 0 \\ b + c = 1 \end{matrix}$ M1A1

Solve the simultaneous equations and obtain: $a = \dfrac{16}{13}$ A1

$b = \dfrac{24}{13}$ A1

$c = \dfrac{-11}{13}$ A1

$d = \dfrac{16}{13}$ A1

$z = \dfrac{16}{13} + i\dfrac{24}{13}$

$w = \dfrac{-11}{13} + i\dfrac{16}{13}$

c Make x the subject: $x = \dfrac{\lambda - 2 - \lambda y}{2}$ (ǂ) M1A1

Substitute: $\lambda(\dfrac{\lambda - 2 - \lambda y}{2}) + 2y = 4\lambda - 8$ M1

Make y the subject: $y = \dfrac{-\lambda^2 + 10\lambda - 16}{4 - \lambda^2}$ M1A1

Substitute into (ǂ) to find x: $x = \dfrac{-8\lambda + 4}{(2 + \lambda)^2}$ M1A1

i This has a unique solution when $\lambda \neq \pm 2$ R1

Then the solution is $x = \dfrac{-8\lambda + 4}{(2 + \lambda)^2}$, $y = \dfrac{-\lambda^2 + 10\lambda - 16}{4 - \lambda^2}$

ii For $\lambda = 2$, the system becomes $\begin{matrix} 2x + 2y = 0 \\ 2x + 2y = 0 \end{matrix}$ A1

Which has infinitely many solutions. R1

iii For $\lambda = -2$, the system becomes $\begin{matrix} 2x - 2y = -4 \\ -2x + 2y = 16 \end{matrix}$ A1

Which has no solutions. R1

d **i** Multiplying out gives $ax^2 - ax + 2x - 2 - 3x^2 + 2x - ax - 2a = 1$ M1A1

Factorising gives $x^2(a-3)+x(2a-4)+6a-3=0$ M1AG

ii This is a quadratic in x, so consider discriminant M1

$D=(2a-4)^2-4(a-3)(6a-3)=-5a^2+17a-5$ A1

Set discriminant as zero: $-5a^2+17a-5=0$ M1

$a=\dfrac{-17+3\sqrt{21}}{-10}$ A1

$a=\dfrac{17+3\sqrt{21}}{10}$ A1

Exam practice: chapters 1 – 8

1 **P1:** Find the range of values of k for which the equation $3kx^2 + k\sqrt{3}x + 3 = 0$ has two real roots. (6)

2 **P1:** Consider the quadratic functions $f(x) = x^2$ and $g(x) = \frac{1}{2}x^2 - 3x + \frac{1}{2}$.

a Express $g(x)$ in the form $g(x) = a(x-h)^2 + k$. (4)

b Find the coordinates of the vertex of the graph of $y = g(x)$. (1)

c The graph of $y = f(x)$ is transformed into the graph of $y = g(x)$ by a sequence of three transformations. Describe each transformation in turn. (3)

3 **P1:** Consider the functions $f(x) = \frac{1}{3}x + 12$ and $g(x) = x^2$.

a Find the value of $(f \circ g)(2\sqrt{3})$. (2)

b Find an expression for $f^{-1}(x)$. (2)

c Solve the equation $f^{-1} \circ g(x) = 0$. (3)

4 **P2:** Jake and Elisa are given a mathematics problem.

The probability that Jake can solve it is 0.35.

If Jake has solved it, the probability that Elisa can solve it is 0.6, otherwise it is 0.45.

a Draw a tree diagram to illustrate the above situation, showing clearly the probabilities on each branch. (3)

b Find the probability that neither student solves the problem. (2)

c Find the probability that at least one of the students can solve the problem. (2)

d Find the probability that Jake solves the problem, given than Elisa has. (4)

5 **P1:** In the binomial expansion of $\left(2x^3 - \frac{1}{x}\right)^8$, find the coefficient of the term containing x^{12}.

(6)

6 P1: Consider the function $f(x)=\dfrac{2}{3x-1}$, $x\neq\dfrac{1}{3}$, $x\in\mathbb{R}$.

a State the range of f . (1)

b Sketch the graph of $y=f(x)$, stating clearly the equations of any asymptotes.

State also the coordinates of any points of intersection with the x and y axes. (5)

c Find the inverse function $f^{-1}(x)$. (3)

7 P2: Ben practises playing the Oboe daily.

The time (in minutes) he spends on daily practice over 28 days is as follows.

10, 15, 30, 35, 40, 40, 45, 55, 60, 62, 64, 64, 66, 68, 70,
70, 72, 75, 75,80, 82,84, 90, 90, 105, 110, 120, 180.

a Find the median time (2)
b Find the lower quartile (2)
c Find the upper quartile (2)
d Find the range (2)
e Determine whether there are any outliers in the data (4)
f Draw a box and whisker diagram for the above data, marking any outliers as required. (3)

8 P2: Neeve conducts a test to determine if there is any correlation between a person's age and the number of hours they watch television per week.

Age	8	42	17	81	45	14	39	42	31	40	28	24
No. of hours	20	15	30	2	25	28	19	14	16	21	26	20

a Find Pearson's product-moment correlation coefficient (r) for this data. (2)

b Interpret the value of Pearson's product-moment correlation coefficient (r) in the context of the question. (1)

c Find the equation of the y on x regression line. (2)

d Using your result from part **c**, determine the number hours per week that a 60-year-old might be expected to watch television. (2)

From her data, Neeve concludes that a person's age affects the number of hours per week that they tend to watch television.

e Explain whether or not this is a valid conclusion. If not, suggest what conclusion Neeve could draw from these results. (3)

9 P2: Consider the function $f(x)=x-4\sqrt{x}$, $x\geq 0$.

a Find an expression for $f'(x)$. (2)

b Find an expression for $f''(x)$. (1)

c Hence, find the coordinates of any turning point(s) on the curve and determine their nature. (5)

d Sketch the graph of $y = f(x)$, showing clearly the coordinates of any turning points and intersections with the axes. (6)

10 P2: a Sketch a graph of the region bounded by the curves $y = e^{\frac{x}{2}}$ and $y = 10 - x^2$. (3)

b Write down a definite integral representing the area of the region drawn. (3)

c Find the size of the bounded area using a GDC. (1)

11 P1: a Find the exact value of (i) $\sin 75^\circ$ (ii) $\cos 75^\circ$. (8)

b Hence find the exact value of $\cos 37.5^\circ$. (5)

12 P1: Use L'Hôpital's rule to find

a $\lim_{x\to 0}\dfrac{\sin(\tan(3x))}{2x}$ (5)

b $\lim_{x\to 0}\dfrac{\ln(x+1)}{\sqrt{x}}$ (5)

At each stage, you should justify that the conditions for L'Hopital's rule are satisfied.

13 P1: All functions in this question map real numbers to real numbers.

a The function $f(x) = ax^2 + bx + c$ is an even function.
Find the value of the constant b. (3)

b The function $g(x) = p\sin x + q\cos x + r\tan x$ is an odd function.
Find the value of the constant q. (3)

c The function $k(x)$ is an odd function. Find the value of $k(0)$. (3)

d The function $h(x)$ is both an odd function and an even function; find $h(x)$. (3)

14 P1: a Prove by induction that 7 divides $8^n - 1$ exactly, for $n \in \mathbb{Z}^+$. (9)

b 2^{300} grains of rice are to be shared equally between 7 famine relief agencies. After each agency has received exactly the same number of grains of rice, find the number of grains of rice that will be left over. (3)

15 P1: a Use row reduction (the Gaussian method to solve the following set of linear equations.

$x + y + z = 10$
$x + 2y + z = 13$ (5)
$x + 2y + 2z = 18$

b Use row reduction to find the values of c for which the following set of linear equations have no solutions.

$x + y + z = 10$
$x + 2y + z = 13$ (4)
$2x + 3y + 2z = c$

c For the value of c in part (**b** which *does* give solutions, find these solutions. (4)

16 P2: **a** Use differentiation to find the first 5 terms in the Maclaurin expansion of $f(x) = (1+x)^{\frac{1}{2}}$.
(8)

b Hence find a rational approximation to $\sqrt{2}$. (3)

c Find the absolute percentage error made when using this rational approximation.

(2)

17 P1: Use implicit differentiation to find the exact equation of the tangent to the curve $25x^2 + 16y^2 = 400$ at the point $\left(\frac{12}{5}, 4\right)$. (8)

18 P2: The hour hand on a clock is 10 cm long and the minute hand is 20 cm long. The clock starts at 12:00. The angle between the hour hand and the minute hand is θ and for more than half an hour θ is increasing at a constant rate of $\frac{11}{12} \times 2\pi$ radians per hour. Let x be the distance between the end of the hour hand and the end of the minute hand. Find the rate at which x is increasing, in cm per hour, when $\theta = \frac{\pi}{6}$. (9)

19 P1: **a** Find $\int \frac{1}{1+v^2} dv - \int \frac{v}{1+v^2} dv$. (2)

b Show that the solution to the differential equation $\frac{dy}{dx} = \frac{x+y}{x-y}$, $x > 0$ is

$\arctan \frac{y}{x} + \ln\left(\frac{1}{\sqrt{x^2+y^2}}\right) = c$. (12)

20 P3: In this question, you will investigate the definite integral $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx \quad n \in \mathbb{N}$.

Your task will be to find an exact formula for it.

a Find

i $\int 1 \, dx$ and hence

ii the exact value of $I_0 = \int_0^{\frac{\pi}{2}} 1\, dx$. (2)

b Find

i $\int \sin x\, dx$ and hence

ii the exact value of $I_1 = \int_0^{\frac{\pi}{2}} \sin x\, dx$. (2)

c Find the exact value of $I_2 = \int_0^{\frac{\pi}{2}} \sin^2 x\, dx$, by employing a double angle formula. (2)

d Find the exact value of $I_3 = \int_0^{\frac{\pi}{2}} \sin^3 x\, dx$, by using the fact that $\sin^2 x = 1 - \cos^2 x$.(3)

e **i** Use the calculator to find I_4 , giving the answer to 10 decimal places.

ii Suggest what fraction $\frac{I_4}{\pi}$ could be, and hence suggest what the exact value of I_4 is. (2)

f Use the calculator to find I_5 , and convert your answer to a fraction. (1)

You will now attempt to find an iterative formula for I_n .

g Write I_{n+2} in the form $\int_0^{\frac{\pi}{2}} \sin^n x \sin^2 x\, dx$ and use $\sin^2 x = 1 - \cos^2 x$ to write I_{n+2} as the difference of two integrals. Then apply integration by parts to the second integral, choosing $\cos x$ as the term to be differentiated, to prove that

$$I_{n+2} = \frac{n+1}{n+2} I_n \ .$$ (7)

h Hence find the exact values of

i I_6

ii I_7 . (2)

You will now attempt to find an explicit formula for I_n .

You will start when n is odd. $I_{2n+1} = \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \frac{2n-4}{2n-3} \times \ldots. \times \frac{4}{5} \times \frac{2}{3} \times 1$.

i By multiplying this expression by $2n(2n-2)(2n-4) \times \ldots \times 4 \times 2$ top and bottom, show that $I_{2n+1} = \frac{2^{2n}(n!)^2}{(2n+1)!}$ (4)

j By applying the same method for the even case, show that $I_{2n} = \dfrac{(2n)!\pi}{2^{2n+1}(n!)^2}$. (4)

Thus the original problem set has been solved.

k Finally, write down $\lim\limits_{n\to\infty} I_n$. (1)

Answers

1 Two real roots implies $b^2 - 4ac > 0$ M1

$\left(k\sqrt{3}\right)^2 - 4(3k)(3) > 0$ A1

$3k^2 - 36k > 0$

$k^2 - 12k > 0$

$k(k-12) > 0$ M1

Critical values are $k = 0$ and $k = 12$ A1

Solution is $k < 0$ or $k > 12$ A1A1

2 **a** $g(x) = \frac{1}{2}x^2 - 3x + \frac{1}{2}$

$= \frac{1}{2}\left[x^2 - 6x + 1\right]$ M1

$= \frac{1}{2}\left[(x-3)^2 - 9 + 1\right]$ M1A1

$= \frac{1}{2}\left[(x-3)^2 - 8\right]$

$= \frac{1}{2}(x-3)^2 - 4$ A1

b $(3, -4)$ A1

c Horizontal translation 3 units to the right A1

Vertical stretch scale factor $\frac{1}{2}$ A1

Vertical translation down 4 units A1

3 **a** $(f \circ g)\left(2\sqrt{3}\right) = f(12)$ M1

$= 16$ A1

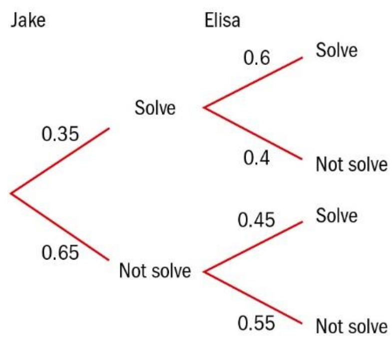
b $y = \frac{1}{3}x + 12 \Rightarrow x = 3(y - 12)$ M1

So $f^{-1}(x) = 3(x - 12)$ A1

c $f^{-1} \circ g(x) = 3\left(x^2 - 12\right)$ M1A1

$f^{-1} \circ g(x) = 0 \Rightarrow x = \pm 2\sqrt{3}$ A1

4 a



M1A1A1

b $0.65 \times 0.55 = 0.3575$ M1A1

c $1-(0.65 \times 0.55) = 0.6425$ M1A1

d $\dfrac{P(\text{Jake and Elisa solve})}{P(\text{Elisa solve})} = \dfrac{0.35 \times 0.6}{(0.35 \times 0.6)+(0.65 \times 0.45)}$ M1A1A1

$= 0.418(3sf)$ A1

5 The general term in the expansion is ${}^8C_r\left(2x^3\right)^r\left(-\dfrac{1}{x}\right)^{8-r}$ M1

$= (-1)^{8-r}\ {}^8C_r\left(\dfrac{2^r x^{3r}}{x^{8-r}}\right)$

$= (-1)^{8-r}\ {}^8C_r\left(2^r x^{4r-8}\right)$ A1

Therefore $4r-8=12$ M1

So $r=5$ A1

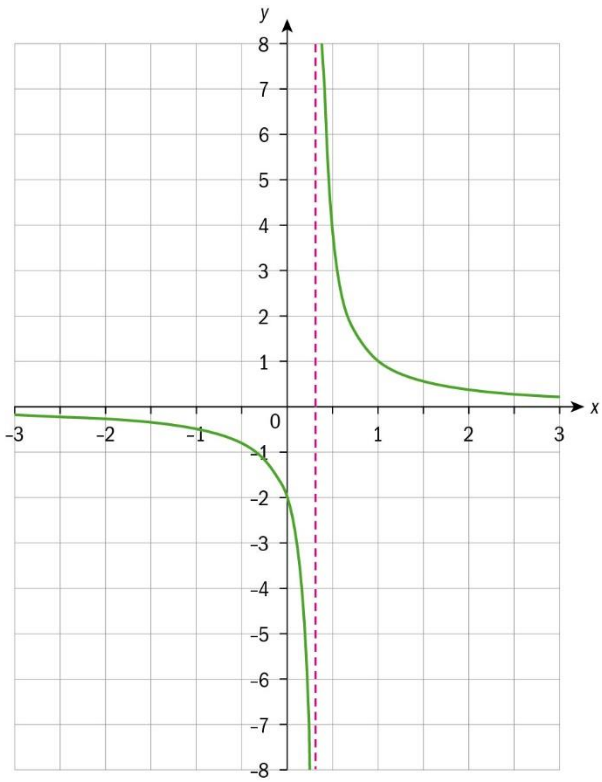
So the term required is ${}^8C_5\left(2x^3\right)^5\left(-\dfrac{1}{x}\right)^3 = -56 \times 2^5 \times x^{12}$

The coefficient is therefore -56×2^5 A1

$= -1792$ A1

6 a Range is $f(x) \neq 0$, $(f(x) \in \mathbb{R})$ A1

b



M1A1A1

Asymptote $x = \frac{1}{3}$ A1

Intersects $y-$ axis at $(0,-2)$ A1

c $y = \frac{2}{3x-1}$

$y(3x-1) = 2$ M1

$3xy - y = 2$

$3xy = 2 + y$

$x = \frac{2+y}{3y}$

$f^{-1}(x) = \frac{2+x}{3x}, x \neq 0$ A1A1

7 **a** 69 minutes M1A1

b 50 minutes M1A1

c 83 minutes M1A1

d 170 minutes M1A1

e Interquartile range is $83-50=33$ A1

$Q_3+1.5\times \text{IQ range}=83+1.5\times 33$ M1

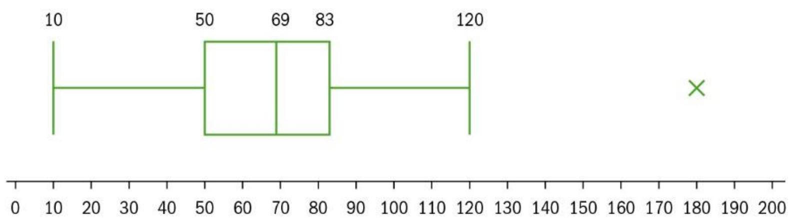
$=132.5$

$Q_1-1.5\times \text{IQ range}=50-1.5\times 33$ M1

$=0.5$

Therefore 180 is an outlier A1

f



M1A1A1

8 **a** Use of GDC to give M1

$r=-0.775$ A1

b This is a strong negative correlation. i.e. as a person's age increases,

the number of hours they watch TV decreases. R1

c Use of GDC to give M1

$y=-0.306x+30.1$ A1

d $y=-0.306\times 60+30.1$

M1

$=11.7$ hours A1

e Neeve is incorrect. A1

A value of $r=-0.775$ indicates a strong negative correlation between

a person's age and the hours per week they watch TV. R1

However, you cannot say this is *causal*. R1

(i.e. You cannot conclude that your age *affects* the amount of TV you watch.)

9 **a** $f(x)=x-4x^{\frac{1}{2}}$

$f'(x)=1-2x^{-\frac{1}{2}}$ M1A1

b $f''(x) = x^{-\frac{3}{2}}$ A1

c Attempting to solve $f'(x) = 0$ M1

$1 - 2x^{-\frac{1}{2}} = 0$

$1 - \frac{2}{\sqrt{x}} = 0$

$x = 4$ A1

$f(4) = 4 - 4\left(4^{\frac{1}{2}}\right) = -4$ A1

Therefore $(4, -4)$ is the only turning point

$f''(4) = 4^{-\frac{3}{2}} = \frac{1}{4^{\frac{3}{2}}} = \frac{1}{8}$ A1

$\frac{1}{8} > 0$. Therefore it is a minimum point R1

d Intersects $y-$ axis at $(0,0)$ A1

Intersects $x-$ axis where $x - 4x^{\frac{1}{2}} = 0$ M1

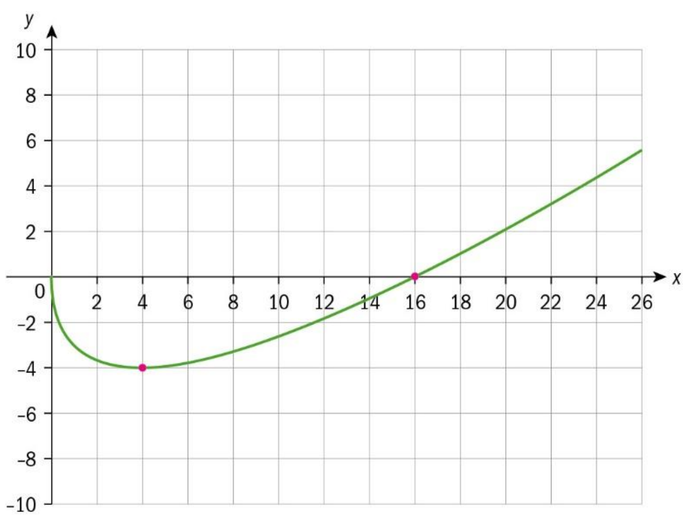
$x\left(1 - 4x^{-\frac{1}{2}}\right) = 0$ M1

$1 - 4x^{-\frac{1}{2}} = 0$ or $x = 0$

$1 - \frac{4}{\sqrt{x}} = 0$

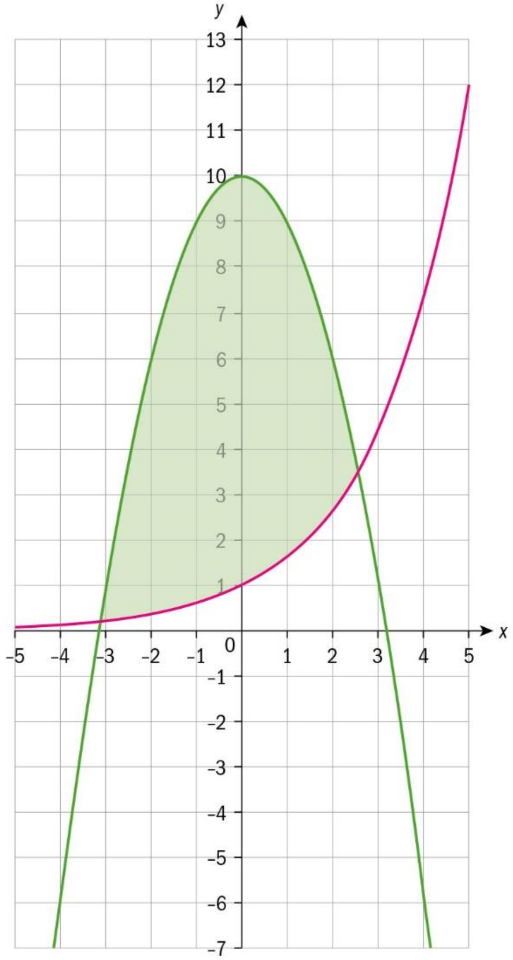
$x = 16$ A1

So crosses $x-$ axis at $(16, 0)$



A1A1

10 a



M1A1A1

b $\int_{-3.129}^{2.538} \left(10 - x^2 - e^{\frac{x}{2}}\right) dx$

Correct limits A1

Correct expression A1

All correct A1

c 34.3 A1

11 a **i** $\sin 75^\circ = \sin(30+45) = \sin 30 \cos 45 + \cos 30 \sin 45$ M1A1

$\frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} = \frac{1+\sqrt{3}}{2\sqrt{2}}$ A1A1

ii $\cos 75^\circ = \cos(30+45) = \cos 30 \cos 45 - \sin 30 \sin 45$ M1A1

$\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}-\frac{1}{2}\times\frac{1}{\sqrt{2}}=\frac{\sqrt{3}-1}{2\sqrt{2}}$ A1A1

b $\cos^2 37.5=\frac{1+\cos 75}{2}=\frac{1+\frac{\sqrt{3}-1}{2\sqrt{2}}}{2}=\frac{2\sqrt{2}+\sqrt{3}-1}{4\sqrt{2}}$

M1A1

$\cos 37.5=\frac{\sqrt{2\sqrt{2}+\sqrt{3}-1}}{2\sqrt[4]{2}}$ where we take the positive square-root as we are in the 1st quadrant M1A1R1

12 a Indeterminate form $\frac{0}{0}$ R1 A1

limit equals $\lim_{x\to 0}\frac{\cos(\tan(3x))\sec^2(3x)\times 3}{2}=\frac{3}{2}$ M1A1A1

b Indeterminate form $\frac{0}{0}$ R1A1

limit equals $\lim_{x\to 0}\frac{\frac{1}{1+x}}{\frac{1}{2\sqrt{x}}}=\lim_{x\to 0}\frac{2\sqrt{x}}{1+x}=0$ M1A1A1

13 a $f(x)=f(-x)\Rightarrow ax^2+bx+c=ax^2-bx+c\Rightarrow 2bx=0\Rightarrow b=0$ M1A1A1

b $g(x)=-g(-x)\Rightarrow p\sin x+q\cos x+r\tan x=-(p\sin(-x)+q\cos(-x)+r\tan(-x))$

$\Rightarrow p\sin x+q\cos x+r\tan x=p\sin x-q\cos x+r\tan x\Rightarrow 2q\cos x=0\Rightarrow q=0$ M1A1A1

c $k(0)=-k(-0)=-k(0)\Rightarrow 2k(0)=0\Rightarrow k(0)=0$ M1A1A1

d $h(x)=h(-x)=-h(-x)\Rightarrow 2h(-x)=0\Rightarrow h(-x)=0,\forall x$

So $h(x)=0,\forall x$ M1A1A1

14 a Let $P(n)$ be the statement that 7 exactly divides 8^n-1

$8^1-1=7=7\times 1$, so $P(1)$ is true. M1A1

Assume that $P(k)$ is true and attempt to prove $P(k+1)$ is true M1

$8^k-1=7A$ for $A\in\mathbb{Z}$ R1

$8^{k+1}-1=8(8^k)-1=8(7A+1)-1=7(8A)+7=7(8A+1)$ M1A1A1

$8A+1\in\mathbb{Z}$ so 7 exactly divides $8^{k+1}-1$ R1

$P(1)$ is true and $P(k)$ true implies $P(k+1)$ true, so by the principle of Mathematical Induction $P(n)$ is true for $n \in \mathbb{Z}^+$. R1

b $2^{300} = 8^{100} = 7B+1$ by part (**a**, where $B \in \mathbb{Z}$. So there is 1 grain of rice left over.

M1A1A1

15 a $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 1 & 2 & 1 & 13 \\ 1 & 2 & 2 & 18 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & 8 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right)$

M1A1A1A1

$x = 2, y = 3, z = 5$ A1

b $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 1 & 2 & 1 & 13 \\ 2 & 3 & 2 & c \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & c-20 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & c-23 \end{array}\right)$ M1A1A1

So if $c \neq 23$ there are no solutions A1

c $c = 23$ $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $x + y + z = 10, y = 3$ R1A1

Solution is $x + z = 7, y = 3$ A1A1

16 a $f(x) = (1+x)^{\frac{1}{2}}$, $f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{\frac{-1}{2}}$, $f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{2} \times \frac{-1}{2}(1+x)^{\frac{-3}{2}}$, $f''(0) = \frac{-1}{4}$

$f'''(x) = \frac{1}{2} \times \frac{-1}{2} \times \frac{-3}{2}(1+x)^{\frac{-5}{2}}$, $f'''(0) = \frac{3}{8}$

$f''''(x) = \frac{1}{2} \times \frac{-1}{2} \times \frac{-3}{2} \times \frac{-5}{2}(1+x)^{\frac{-7}{2}}$, $f''''(0) = \frac{-15}{16}$ M1A1A1A1A1A1

$f(x) = (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}\frac{x}{1!} - \frac{1}{4}\frac{x^2}{2!} + \frac{3}{8}\frac{x^3}{3!} - \frac{15}{16}\frac{x^4}{4!}\ldots$ M1

$f(x) = (1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128}\ldots$ A1

b Putting $x = 1$ gives $\sqrt{2} \simeq 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} = \frac{179}{128}$ M1A1A1

c $\dfrac{\left|\dfrac{179}{128}-\sqrt{2}\right|}{\sqrt{2}}\times 100\% = 1.12\%(3sf)$ M1A1

17 $50x+32y\dfrac{dy}{dx}=0 \Rightarrow \dfrac{dy}{dx}=-\dfrac{25x}{16y}$ M1A1A1

At $\left(\dfrac{12}{5},4\right)$ $\dfrac{dy}{dx}=-\dfrac{15}{16}$ A1

Tangent is $y=-\dfrac{15}{16}x+c$

Through $\left(\dfrac{12}{5},4\right)$

$4=-\dfrac{15}{16}\times\dfrac{12}{5}+c \Rightarrow c=\dfrac{25}{4}$ R1M1A1

Tangent is $y=-\dfrac{15}{16}x+\dfrac{25}{4}$ A1

18 $x^2=10^2+20^2-2\times10\times20\cos\theta$ M1A1

Differentiating implicitly with respect to t,

$2x\dfrac{dx}{dt}=2\times10\times20\sin\theta\dfrac{d\theta}{dt}$ M1A1

At $\theta=\dfrac{\pi}{6}$, $\dfrac{d\theta}{dt}=\dfrac{11}{12}\times2\pi$ so $x\dfrac{dx}{dt}=200\times\dfrac{1}{2}\times\dfrac{11}{12}\times2\pi=\dfrac{550}{3}\pi$ M1A1

At $\theta=\dfrac{\pi}{6}$ $x^2=10^2+20^2-2\times10\times20\times\dfrac{\sqrt{3}}{2} \Rightarrow x=12.393\ldots$ M1A1

Gives $\dfrac{dx}{dt}=\dfrac{\frac{550}{3}\pi}{12.393\ldots}=46.5\,cm/h(3sf)$ A1

19 a $\arctan v-\dfrac{1}{2}\ln\left(1+v^2\right)+c$ A1A1

b $\dfrac{dy}{dx}=\dfrac{1+\dfrac{y}{x}}{1-\dfrac{y}{x}}$ M1

Use the substitution $v=\dfrac{y}{x}$ R1A1

$y=vx \Rightarrow \dfrac{dy}{dx}=x\dfrac{dv}{dx}+v$ so the equation becomes

$x\dfrac{dv}{dx}+v=\dfrac{1+v}{1-v} \Rightarrow x\dfrac{dv}{dx}=\dfrac{1+v}{1-v}-v$ M1A1

$\Rightarrow x\dfrac{dv}{dx} = \dfrac{1+v^2}{1-v}$ A1

Separating variables and integrating gives $\int \dfrac{1-v}{1+v^2}dv = \int \dfrac{1}{x}dx$ A1

Using result from part (**a** to integrate LHS gives

$\arctan v - \dfrac{1}{2}\ln\left(1+v^2\right) = \ln x + c$ M1A1

Substituting $v = \dfrac{y}{x}$ back in gives

$\arctan\dfrac{y}{x} - \dfrac{1}{2}\ln\left(1+\dfrac{y^2}{x^2}\right) - \ln x = c$ M1

$\arctan\dfrac{y}{x} - \dfrac{1}{2}\ln\left(\dfrac{x^2+y^2}{x^2}\right) - \ln x = c$ A1

$\arctan\dfrac{y}{x} + \ln\left(\dfrac{x}{\sqrt{x^2+y^2}}\right) - \ln x = c$ A1

$\arctan\dfrac{y}{x} + \ln\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) = c$ AG

20 a **i** $\int 1\,dx = x + c$

ii $I_0 = \left[x\right]_0^{\frac{\pi}{2}} = \dfrac{\pi}{2}$ A1A1

b **i** $\int \sin x\,dx = -\cos x + c$

ii $I_1 = \left[-\cos x\right]_0^{\frac{\pi}{2}} = 1$ A1A1

c $I_2 = \int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \int_0^{\frac{\pi}{2}} \dfrac{1-\cos 2x}{2}dx = \left[\dfrac{x}{2} - \dfrac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} = \dfrac{\pi}{4}$ M1A1

d $I_3 = \int_0^{\frac{\pi}{2}} \sin^3 x\,dx = \int_0^{\frac{\pi}{2}} \sin x - \sin x\cos^2 x = \left[-\cos x + \dfrac{\cos^3 x}{3}\right]_0^{\frac{\pi}{2}} = -\left(-1+\dfrac{1}{3}\right) = \dfrac{2}{3}$ M1A1A1

e **i** $I_4 = 0.5890486225(10dp)$

ii $\dfrac{I_4}{\pi}$ could be $\dfrac{3}{16}$,giving $I_4 = \dfrac{3\pi}{16}$ A1A1

f calculator gives $I_5 = \dfrac{8}{15}$ A1

g $I_{n+2} = \int_0^{\frac{\pi}{2}} \sin^n x\,dx - \int_0^{\frac{\pi}{2}} \sin^n x \cos^2 x\,dx$ A1

Integration by parts $\begin{cases} u = \cos x & v = \dfrac{1}{n+1}\sin^{n+1} x \\ \dfrac{du}{dx} = -\sin x & \dfrac{dv}{dx} = \sin^n x \cos x \end{cases}$ M1A1A1

$\int_0^{\frac{\pi}{2}} \sin^n x \cos^2 x\,dx = \left[\frac{1}{n+1}\cos x \sin^{n+1} x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{1}{n+1}\sin^{n+2} x\ dx = 0 + \frac{I_{n+2}}{n+1}$ M1A1

So $I_{n+2} = I_n - \frac{I_{n+2}}{n+1}$ giving $I_{n+2} = \frac{n+1}{n+2} I_n$ A1AG

h **i** $I_6 = \frac{15\pi}{96}$

ii $I_7 = \frac{16}{35}$ A1A1

i Doing the multiplication gives

$$I_{2n+1} = \frac{(2n(2n-2)(2n-4)\times\ldots\times 4\times 2)(2n(2n-2)(2n-4)\times\ldots\times 4\times 2)}{(2n+1)(2n)(2n-1)(2n-2)\times\ldots\times 4\times 3\times 2}$$

M1A1

$= \frac{2^n n!2^n n!}{(2n+1)!} = \frac{2^{2n}(n!)^2}{(2n+1)!}$ as required. R1A1AG

j For the even case $I_{2n} = \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} \times \ldots \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$ M1

Multiplying by the same factor top and bottom gives

$$\frac{2n(2n-1)(2n-2)(2n-3)(2n-4)\times\ldots\times 4\times 3\times 2}{(2n(2n-2)(2n-4)\times\ldots\times 4\times 2)(2n(2n-2)(2n-4)\times\ldots\times 4\times 2)} \times \frac{\pi}{2}$$ M1A1

$= \frac{(2n)!\pi}{2^{2n+1}(n!)^2}$ R1AG

k limit is 0. A1

Exam practice: chapters 1 – 11

1 **P2:** In a geometric series, the second term is 24 and the fifth term is 1.536.

a Find the common ratio for the series. (5)

b Find the first term in the series. (2)

c Find the sum to infinity of the series. (2)

2 **P1:** Find the term in x^3 in the binomial expansion of $(2-x)(3+x)^5$. (5)

3 **P1:** The line $y=k$ is a tangent to the curve $y=e^x(1-x)$. Determine the value of k . (6)

4 **P1:** Given $\int_2^a \frac{10}{5x+4}\,dx = 4\ln 2$, $(a>2)$, find the exact value of a . (7)

5 **P1:** Two functions are given by $f(x)=\frac{5x-3}{2x+6}$, $x\neq -3$,

$x\in\mathbb{R}$ and $g(x)=\sqrt{x}$, $x>0$, $x\in\mathbb{R}$.

a Write down the equation of the vertical asymptote of the graph of $y=f(x)$. (1)

b Write down the equation of the horizontal asymptote of the graph of $y=f(x)$. (1)

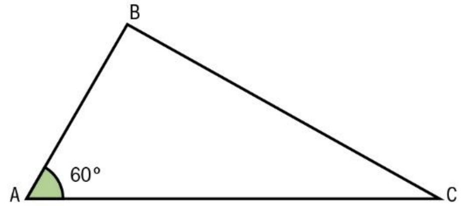
c Sketch the graph of $y=f(x)$. On your sketch, mark the horizontal and vertical asymptotes as dashed lines. (3)

d Write down the range of the function $f(x)$. (1)

e Write down an expression for the function $(g\circ f)(x)$. (1)

f Find the domain of $(g\circ f)(x)$. (2)

6 **P1:** Triangle ABC has sides $AB=4\sqrt{2}-3$, $AC=4\sqrt{2}+3$ and $\angle BAC=60^\circ$.



a Find the exact area of the triangle. (3)

b Find the exact length of BC . (4)

7 P1: Solve the equation $\cos 2\theta = 3\cos\theta - 2$, giving your answers in the range $0 \le \theta < 2\pi$. (7)

8 P2: Approximately 4% of eggs produced and sold by a local farm are cracked.

Jerry buys 24 eggs from the farm.

a Find the probability that exactly two of Jerry's eggs are cracked. (2)
b Find the probability that Jerry buys no more than four cracked eggs. (2)
c Find the probability that Jerry buys at least two cracked eggs. (2)
d Find the variance of the number of cracked eggs. (2)

9 P1: Two events A and B are independent. It is given that $P(A) = 0.3$ and $P(B) = 0.8$.

a State, with a reason, whether events A and B are mutually exclusive. (2)
b Find the probability of the event:

i $A \cap B$

ii $A \cup B$

iii $A \mid B'$

iv $A' \cap B$ (8)

10 P1: By using the substitution $u = 1 + \cos 2x$, show that $\int_0^{\frac{\pi}{3}} \frac{\sin 2x}{1+\cos 2x}\,dx = \ln 2$. (8)

11 P1: Solve the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = \sin x \text{, for } x > 0 \text{, given that } y\left(\frac{\pi}{2}\right) = 0 .$$

Give your answer in the form $y = y(x)$ (11)

12 P1: **a** Use the substitution $\alpha = \frac{\pi}{2} - \beta$ to show that

$$\int_0^{\frac{\pi}{2}} \sin^6\alpha \cos^2\alpha \, d\alpha = \int_0^{\frac{\pi}{2}} \cos^6\beta \sin^2\beta \, d\beta$$ (5)

b Use the substitution $\alpha = \frac{\pi}{2} - \beta$ to find the value of $I = \int_0^{\frac{\pi}{2}} \sin^5\alpha - \cos^5\alpha \, d\alpha$. (6)

13 P2: A committee of 6 teachers is to be formed, to create a new Maths syllabus.
There are 12 teachers, 6 male and 6 female, from which the committee could be made up.

There is a rule that says that the committee cannot be made up of teachers all of the same gender.

a Find the total number of different ways in which the committee can be formed. (3)

One of the male teachers is called Mr Angry.
He will be angry if and only if he is not selected for the committee.

b Find the total number of different ways in which the committee can be formed if it is to include Mr Angry. (3)

c Given that the committee is chosen at random, find the probability that Mr Angry does become angry. (2)

14 P1: The binomial coefficient ${}^nC_r$ and is given by the formula ${}^nC_r = \dfrac{n!}{(n-r)!r!}$.

a Use this formula to prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$. (4)

b Prove by induction on $n \in \mathbb{Z}^+$, that

$${}^nC_r = {}^{n-1}C_r + {}^{n-2}C_{r-1} + {}^{n-3}C_{r-2} + \ldots + {}^{n-r-1}C_0 \quad \text{for } r < n .$$ (8)

15 P1: A point P has the coordinates $(1,-1,1)$. A line L is given by the vector arametric equation $\mathbf{r} = \begin{pmatrix}1\\2\\4\end{pmatrix} + t\begin{pmatrix}1\\1\\1\end{pmatrix}$.

A plane Π is given by the Cartesian equation $2x + y + 2z + 6 = 0$.

a Write down a vector perpendicular to the plane. (1)

b Hence find the coordinates of the point D on the plane that is nearest to the point P.

(5)

c Hence find the perpendicular distance from P to Π. (2)

Let F be the foot of the perpendicular from P to the line L.

d Find the coordinates of the point F. (7)

e Hence find the perpendicular distance from P to L. (2)

f Find the coordinates of the point of intersection of L and Π. (3)

16 P2: A car is travelling in the desert.

It starts at point $(10,20)$ and has constant velocity vector $\begin{pmatrix}5\\12\end{pmatrix}$ km h^{-1}.

The car's position vector at time t (measured in hours) is given by $\mathbf{r} = \begin{pmatrix}10\\20\end{pmatrix} + t\begin{pmatrix}5\\12\end{pmatrix}$.

Distances are measured in kilometres.

a Find the speed of the car. (2)

A police motorcycle starts at the same time as the car, but from the point $(0,0)$.

It has constant velocity vector of $\begin{pmatrix} a \\ b \end{pmatrix}$ km h^{-1} and speed of 20 km h^{-1}.

b Find the velocity vector $\begin{pmatrix} a \\ b \end{pmatrix}$ if the motorcycle is going to intercept the car. (11)

c Find the value of t when the interception happens. (2)

17 P1: Find the exact co-ordinates of the point where the normal to the curve $y = e^{2x}$ at $x = 0$ intersects the tangent to the curve $y = \ln x$ at $x = 1$. (13)

18 P2: The probability distribution function for a continuous random variable X is given by

$f(x) = \frac{a}{x}$ for $1 \le x \le e$

0 otherwise.

a Show that the exact value of a is 1 . (3)

b Find the exact value of

i the mean

ii the variance

iii the median

of X . (10)

c Calculate $P(X \le 2)$, giving the answer to 3 decimal places. (2)

d Calculate $P(X \le 2 | X \ge 1.5)$, giving the answer to 3 decimal places. (3)

19 P2: **a** Use technology to write down the value of i^i , giving the answer to 5 decimal places. (1)

b Write the complex number i in the form $re^{i\theta}$ and hence find the exact value of i^i, the answer in the form $a+bi$. (4)

c Use technology to write down the value of 2^i. Give your answer in the form $a+bi$, correct to 5 decimal places. (2)

d Using the fact that $2 = e^{\ln 2}$, find the exact value of 2^i, giving the answer in the form $a+bi$. (2)

e Using the techniques of parts (a and (b, find an exact expression for $(rcis\theta)^i$ where $r > 0$, in the form $a+bi$. (5)

20 P3: In this question, you will use complex numbers to investigate connections between powers of trigonometrical functions and trigonometrical functions of multiple angles.

Let z be a complex number with modulus of 1 and argument θ , so $z = \cos\theta + i\sin\theta$ (abbreviated to $z = \mathrm{c}i\mathrm{s}\theta$).

a Using De Moivre's theorem, prove that $z^n + \frac{1}{z^n} = 2\cos n\theta, n \in \mathbb{Z}^+$. (6)

b Expand $\left(z + \frac{1}{z}\right)^2$ using the binomial theorem. Apply the result obtained in part **a** to both sides of the equation to show that $\cos^2\theta = \frac{1}{2}\cos 2\theta + \frac{1}{2}$. (5)

c Using the same method as in part (b, expand $\left(z + \frac{1}{z}\right)^3$ and use the result to show that $\cos^3\theta = a\cos 3\theta + b\cos\theta$, where the fractions a and b are to be determined. (Hint: this time, you will have to apply the result from part (a twice. (7)

d Prove that $z^n - \frac{1}{z^n} = 2i\sin n\theta, n \in \mathbb{Z}^+$. (4)

e Use similar techniques as in part **c** to show that $\sin^3\theta = p\sin 3\theta + q\sin\theta$, where the fractions p and q are to be determined. (8)

f Hence find $\int \sin^3\theta \, d\theta$. (3)

Answers

1 **a** $ar = 24$ A1

$ar^4 = 1.536$ A1

Attempt to solve simultaneously M1

$\frac{ar^4}{ar} = \frac{1.536}{24}$

$r^3 = 0.064$ A1

$r = 0.4$ A1

b $a = \frac{24}{r} = \frac{24}{0.4} = 60$ M1A1

c $S_\infty = \frac{a}{1-r}$ M1

$= \frac{60}{1-0.4} = \frac{60}{0.6} = 100$ A1

2 We require $2 \times \binom{5}{3} 3^2 x^3 + (-x) \times \binom{5}{2} 3^3 x^2$ M1A1A1

$= 180x^3 - 270x^3$ A1

$= -90x^3$ A1

3 $y = k$ must intersect $y = e^x(1-x)$ at a maximum or minimum point. R1

$\frac{dy}{dx} = -e^x + (1-x)e^x = -xe^x$ M1A1

$\frac{dy}{dx} = 0 \Rightarrow x = 0$ M1A1

At $x = 0$, $y = 1$ A1

So $k = 1$

4 $\int_2^a \frac{10}{5x+4} dx = [2\ln(5x+4)]_2^a$ M1A1

$= 2\ln(5a+4) - 2\ln 14$ M1

$= 2\ln\left(\frac{5a+4}{14}\right)$ M1

$4\ln 2 = 2\ln 4$ A1

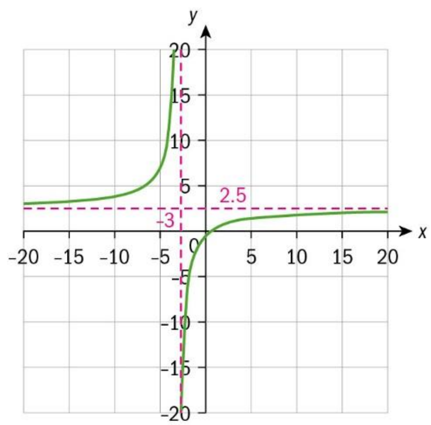
$\frac{5a+4}{14} = 4$ M1

$a = \frac{52}{5}$ A1

5 **a** $x = -3$ A1

b $y = \frac{5}{2}$ A1

c



left-hand branch A1

right-hand branch A1

asymptotes A1

d $f(x) \neq \frac{5}{2}$, $(f(x) \in \mathbb{R})$ A1

e $g \circ f(x) = \sqrt{\frac{5x-3}{2x+6}}$ A1

f $x < -3$, $x \geq \frac{3}{5}$, $(x \in \mathbb{R})$ A1A1

6 **a** Attempt to use sine rule for area M1

Area $= \frac{1}{2} \times (4\sqrt{2}-3) \times (4\sqrt{2}+3) \times \sin 60^\circ$ A1

$= \frac{1}{2} \times 23 \times \frac{\sqrt{3}}{2}$

$= \frac{23\sqrt{3}}{4}$ A1

b Attempt to use cosine rule M1

$AC^2 = (4\sqrt{2}-3)^2 + (4\sqrt{2}+3)^2 - 2(4\sqrt{2}-3)(4\sqrt{2}+3)\cos 60^\circ$ A1

$= 32 + 9 - 24\sqrt{2} + 32 + 9 + 24\sqrt{2} - 2 \times 23 \times \frac{1}{2}$ A1

$= 82 - 23$

$= 59$

So $AC = \sqrt{59}$ A1

7 $2\cos^2\theta - 1 = 3\cos\theta - 2$ M1

$2\cos^2\theta - 3\cos\theta + 1 = 0$

$(2\cos\theta - 1)(\cos\theta - 1) = 0$ M1A1

$\cos\theta = \frac{1}{2}$ or $\cos\theta = 1$ A1

$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \theta = \frac{5\pi}{3}$ A1A1

$\cos\theta = 1 \Rightarrow \theta = 0$ A1

8 **a** Let X represent the number of cracked eggs Jerry buys.

$X \sim B(24, 0.04)$

$P(X = 2) = {}^{24}C_2 (0.04)^2 (0.96)^{22}$ or uses GDC directly M1

$= 0.180$ A1

b $P(X \le 4) = 0.998$ M1A1

c $P(X \ge 2) = 0.249$ M1A1

d $\text{Var}(X) = np(1 - p)$ M1

$= 24 \times 0.04 \times 0.96$

$= 0.9216$ A1

9 **a** If they were mutually exclusive, then $P(A \cap B) = 0$, A1

but since they are independent, we have $P(A \cap B) = P(A)P(B) \neq 0$.

Therefore we have a contradiction, and so A and B are not mutually exclusive. R1

b **i** $P(A \cap B) = P(A)P(B) = 0.3 \times 0.8 = 0.24$ M1A1

ii $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.8 - 0.24 = 0.86$ M1A1

iii $P(A \mid B') = \dfrac{P(A \cap B')}{P(B')} = \dfrac{P(A) - P(A \cap B)}{P(B')} = \dfrac{0.3 - 0.24}{0.2} = \dfrac{0.06}{0.2} = 0.3$ M1A1

iv $P(A' \cap B) = P(B) - P(A \cap B) = 0.8 - 0.24 = 0.56$ M1A1

10

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = -2\sin 2x$ A1

$x = \dfrac{\pi}{3} \Rightarrow u = \dfrac{1}{2}$ and $x = 0 \Rightarrow u = 2$ A1

$\displaystyle\int_0^{\frac{\pi}{3}} \frac{\sin 2x}{1 + \cos 2x}\,\mathrm{d}x = -\frac{1}{2}\int_2^{\frac{1}{2}} \frac{\mathrm{d}u}{u}$ M1A1

$= -\dfrac{1}{2}[\ln u]_2^{\frac{1}{2}}$ A1

$= -\dfrac{1}{2}\left(\ln\dfrac{1}{2} - \ln 2\right)$ M1

$= -\dfrac{1}{2}(-\ln 2 - \ln 2)$ A1

$= \dfrac{1}{2}(2\ln 2)$ A1

$= \ln 2$ AG

11 1st order linear. Integrating factor is $e^{\int \frac{1}{x}dx} = e^{\ln x} = x$ M1A1A1

$x\dfrac{dy}{dx} + y = x\sin x \Rightarrow \dfrac{d(xy)}{dx} = x\sin x \Rightarrow xy = \int x\sin x\,dx$ M1A1

Integration by parts $\begin{cases} u = x & v = -\cos x \\ \dfrac{du}{dx} = 1 & \dfrac{dv}{dx} = \sin x \end{cases}$ M1A1

$xy = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + c$ A1

Through $\left(\dfrac{\pi}{2}, 0\right) \Rightarrow 0 = 0 + 1 + c \Rightarrow c = -1$, $xy = -x\cos x + \sin x - 1$ M1A1

$y = -\cos x + \dfrac{\sin x - 1}{x}$ A1

12 a $\frac{d\alpha}{d\beta} = -1$

$$\int_0^{\frac{\pi}{2}} \sin^6\alpha \cos^2 \alpha \, d\alpha = \int_{\frac{\pi}{2}}^{0} \sin^6(\frac{\pi}{2} - \beta) \cos^2(\frac{\pi}{2} - \beta) \times -1 \, d\beta$$ A1M1A1

$$-\int_{\frac{\pi}{2}}^{0} \cos^6\beta \sin^2 \beta \, d\beta = \int_0^{\frac{\pi}{2}} \cos^6\beta \sin^2 \beta \, d\beta$$ M1A1AG

b $$I = \int_0^{\frac{\pi}{2}} \sin^5\alpha - \cos^5 \alpha \, d\alpha = \int_{\frac{\pi}{2}}^{0} \sin^5(\frac{\pi}{2} - \beta) - \cos^5(\frac{\pi}{2} - \beta) \times -1 d\beta$$ M1A1

$$= \int_0^{\frac{\pi}{2}} \cos^5\beta - \sin^5 \beta \, d\beta = -I$$ M1A1

$2I = 0 \Rightarrow I = 0$ M1A1

13 a Choosing 6 from 12 without all male or all female is ${}^{12}C_6 - 2 = 922$ M1A1A1

b Choosing Mr Angry, then choosing 5 from 11 without all male is ${}^{11}C_5 - 1 = 461$ M1A1A1

c $1 - \frac{461}{922} = 0.5$

M1A1

14 a $${}^nC_r + {}^nC_{r-1} = \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!} = \frac{n!(n-r+1+r)}{(n-r+1)!r!}$$ M1A1

$$\frac{n!(n+1)}{(n+1-r)!r!} = \frac{(n+1)!}{((n+1)-r)!r!} = {}^{n+1}C_r$$ M1A1AG

b Let $P(n)$ be the statement ${}^nC_r = {}^{n-1}C_r + {}^{n-2}C_{r-1} + {}^{n-3}C_{r-2} + \ldots + {}^{n-r-1}C_0$.

For $n = 1$, LHS must be ${}^1C_0$ which is 1, RHS is ${}^0C_0$ which is also 1, so $P(1)$ is true.

M1A1

We assume the result for $P(k)$ and attempt to prove for $P(k+1)$. M1

$${}^kC_r = {}^{k-1}C_r + {}^{k-2}C_{r-1} + {}^{k-3}C_{r-2} + \ldots + {}^{k-r-1}C_0$$

$$\Rightarrow {}^kC_r + {}^kC_{r+1} = {}^kC_{r+1} + {}^{k-1}C_r + {}^{k-2}C_{r-1} + {}^{k-3}C_{r-2} + \ldots + {}^{k-r-1}C_0$$ M1A1

Using (a ${}^{k+1}C_{r+1} = {}^kC_{r+1} + {}^{k-1}C_r + {}^{k-2}C_{r-1} + {}^{k-3}C_{r-2} + \ldots + {}^{(k+1)-(r+1)-1}C_0$ R1A1

Which is statement $P(k+1)$ but with an $r+1$ instead of r .(and $r+1 < k+1$)

Since $P(0)$ is true and $P(k)$ true implies $P(k+1)$ true then by the principle of Mathematical Induction $P(n)$ is true for $n \in \mathbb{Z}^+$. R1

15 a $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ A1

b $\overrightarrow{OD} = \overrightarrow{OP} + \overrightarrow{PD} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+2\lambda \\ -1+\lambda \\ 1+2\lambda \end{pmatrix}$. M1A1

D lies in plane $\Rightarrow 2(1+2\lambda)+1(-1+\lambda)+2(1+2\lambda)+6=0 \;\Rightarrow 9\lambda+9=0 \Rightarrow \lambda=-1$

M1A1

$D=(-1,-2,-1)$ A1

c distance is $\sqrt{2^2+1^2+2^2}=3$ M1A1

d $\overrightarrow{PF} = -\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1+t \\ 2+t \\ 4+t \end{pmatrix} = \begin{pmatrix} t \\ 3+t \\ 3+t \end{pmatrix}$ Direction of line is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ M1A1 A1

Require these to be perpendicular, taking the dot product R1

$1(t)+1(3+t)+1(3+t)=0 \Rightarrow t=-2$
M1A1

$F=(-1,0,2)$

A1

e distance is $\sqrt{2^2+1^2+1^2}=\sqrt{6}$ M1A1

f require $2(1+t)+1(2+t)+2(4+t)+6=0 \Rightarrow t=\frac{-18}{5}$ M1A1

Intersection point is $(-2.6,-1.6,0.4)$ A1

16 a $\sqrt{5^2+12^2}=13$ km h^{-1} M1A1

b Path of motorcycle is given by $\mathbf{s}=t\begin{pmatrix} a \\ b \end{pmatrix}$ M1A1

$\sqrt{a^2+b^2}=20 \Rightarrow a^2+b^2=400$ M1A1

Require $\begin{pmatrix} 10 \\ 20 \end{pmatrix} + t\begin{pmatrix} 5 \\ 12 \end{pmatrix} = t\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow 10=t(a-5)\;,20=t(b-12)$ M1A1

$2=\frac{b-12}{a-5} \Rightarrow b=2a+2$ M1

$a^2+(2a+2)^2=400 \Rightarrow 5a^2+8a-396=0 \;\Rightarrow a=8.1353\ldots, b=18.2706\ldots$ M1A1

$\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 8.14 \\ 18.3 \end{pmatrix}$ km h^{-1} (3*sf*) A1A1

c $t=\dfrac{10}{3.1353}=3.19h(3sf)$ M1A1

17 Find normal to $y=e^{2x}$ at $x=0$:

$\dfrac{dy}{dx}=2e^{2x}$, so at $x=0$, $\dfrac{dy}{dx}=2$. Eqn of normal is $y=-\dfrac{1}{2}x+c$ M1A1A1

Through $(0,1) \Rightarrow y=-\dfrac{1}{2}x+1$ M1A1

Find tangent to $y=\ln x$ at $x=1$:

$\dfrac{dy}{dx}=\dfrac{1}{x}$, so at $x=1$, $\dfrac{dy}{dx}=1$. Eqn of tangent is $y=x+c$ M1A1A1

Through $(1,0) \Rightarrow y=x-1$ M1A1

$-\dfrac{1}{2}x+1=x-1 \Rightarrow x=\dfrac{4}{3} \Rightarrow y=\dfrac{1}{3}$ M1A1A1

18 a require $\displaystyle\int_1^e \frac{a}{x}dx=1 \Rightarrow a[\ln x]_1^e=a\ln e-a\ln 1=a=1$ R1M1A1AG

b **i** $\displaystyle\mu=\int_1^e \frac{x}{x}dx=\int_1^e 1\,dx=[x]_1^e=e-1$ M1A1A1

ii $\displaystyle\sigma^2=\int_1^e \frac{x^2}{x}dx-(e-1)^2=\int_1^e x\,dx-(e-1)^2=\left[\frac{x^2}{2}\right]_1^e-e^2+2e-1$ M1A1A1

$=\dfrac{e^2}{2}-\dfrac{1}{2}-e^2+2e-1=2e-\dfrac{e^2}{2}-\dfrac{3}{2}$ A1

iii $\displaystyle\int_1^M \frac{1}{x}dx=\frac{1}{2} \Rightarrow [\ln x]_1^M=\ln M=\frac{1}{2} \Rightarrow M=e^{\frac{1}{2}}$ M1A1A1

c $\displaystyle P(X\le 2)=\int_1^2 \frac{1}{x}dx=0.693$ M1A1

d $P(X\le 2|X\ge 1.5)=\dfrac{P(1.5\le X\le 2)}{P(1.5\le X)}=\dfrac{\int_{1.5}^{2}\frac{1}{x}dx}{\int_{1.5}^{e}\frac{1}{x}dx}=0.484$ M1A1A1

19 a 0.20788 A1

b $i = 1e^{i\frac{\pi}{2}}$, so $i^i = e^{i^2\frac{\pi}{2}} = e^{-\frac{\pi}{2}} = e^{-\frac{\pi}{2}} + 0i$ M1A1 M1A1

c $0.76924 + 0.63896i$ A1A1

d $2^i = e^{i\ln 2} = cis(\ln 2) = \cos(\ln 2) + \sin(\ln 2)i$ M1A1

e $(rcis\theta)^i = \left(re^{i\theta}\right)^i = \left(e^{\ln r}e^{i\theta}\right)^i = e^{i\ln r}e^{-\theta} = e^{-\theta}cis(\ln r)$ M1M1A1

$= e^{-\theta}\cos(\ln r) + e^{-\theta}\sin(\ln r)i$ A1A1

20 a $z^n + \dfrac{1}{z^n} = (cis\theta)^n + (cis\theta)^{-n} = cisn\theta + cis(-n\theta)$ M1A1A1

$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta + i\sin n\theta + \cos(n\theta) - i\sin(n\theta)$ M1R1A1

$= 2\cos n\theta.$ AG

b $\left(z + \dfrac{1}{z}\right)^2 = z^2 + 2 + \dfrac{1}{z^2} = z^2 + \dfrac{1}{z^2} + 2 \Rightarrow$ M1A1

$(2\cos\theta)^2 = 2\cos 2\theta + 2 \Rightarrow 4\cos^2\theta = 2\cos 2\theta + 2 \Rightarrow$ M1A1A1

$\cos^2\theta = \dfrac{1}{2}\cos 2\theta + \dfrac{1}{2}$ AG

c $\left(z + \dfrac{1}{z}\right)^3 = z^3 + 3z + 3\dfrac{1}{z} + \dfrac{1}{z^3} = z^3 + \dfrac{1}{z^3} + 3\left(z + \dfrac{1}{z}\right) \Rightarrow$ M1A1A1

$(2\cos\theta)^3 = 2\cos 3\theta + 3\cos\theta \Rightarrow 8\cos^3\theta = 2\cos 3\theta + 6\cos\theta \Rightarrow$ M1A1

$\cos^3\theta = \dfrac{1}{4}\cos 3\theta + \dfrac{3}{4}\cos\theta$ A1A1

d $z^n - \dfrac{1}{z^n} = (cis\theta)^n - (cis\theta)^{-n} = cisn\theta - cis(-n\theta)$ M1A1

$= \cos n\theta + i\sin n\theta - \cos(-n\theta) - i\sin(-n\theta) = \cos n\theta + i\sin n\theta - \cos(n\theta) + i\sin(n\theta)$ M1A1

$= 2i\sin n\theta.$ AG

e $\left(z - \dfrac{1}{z}\right)^3 = z^3 - 3z + 3\dfrac{1}{z} - \dfrac{1}{z^3} = z^3 - \dfrac{1}{z^3} - 3\left(z - \dfrac{1}{z}\right) \Rightarrow$ M1A1A1

$(2i\sin\theta)^3 = 2i\sin 3\theta - 3(2i\sin\theta) \Rightarrow -8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta \Rightarrow$ M1A1A1

$\sin^3\theta = \dfrac{-1}{4}\sin 3\theta + \dfrac{3}{4}\sin\theta$ A1A1

f $\int \sin^3\theta\, d\theta = \int \dfrac{-1}{4}\sin 3\theta + \dfrac{3}{4}\sin\theta\, d\theta = \dfrac{1}{12}\cos 3\theta - \dfrac{3}{4}\cos\theta + c$ M1A1A1